

SECTION 6

REACTOR PHYSICS

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6-1 REACTOR THEORY

BY

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The principal object of reactor theory is to provide useful techniques which will give an accurate description of the spatial and energy distributions of neutrons within a chain-reacting system. In a system operating in the steady state it is required that for a given source of neutrons born in fission the slowing down and eventual capture or loss of these neutrons shall be correctly described. The resulting distribution of fissions induced by these neutron captures must agree both in strength and in distribution with the given source. Only if this condition is satisfied will the reactor be critical and operating in an equilibrium condition.

In order to be of use to engineers, reactor theory must obtain this correct description of the neutron distribution for reactor geometries of engineering interest and must include a realistic representation of the complex nuclear cross sections of the materials of which the system is composed. It often turns out, however, that these practical considerations prevent the accurate mathematical descriptions from being brought to a solution with any reasonable amount of effort. Consequently, the subject of reactor theory is largely composed of methods which may be used to obtain approximate solutions to a detailed problem for various special cases of engineering interest. This chapter is devoted, then, to a discussion of these approximations and their validity in order to present an over-all viewpoint to the nuclear engineer. Detailed mathematical treatment is left to the references.

In choosing among the various available methods, one should consider several specific questions:

Is it possible to use some rough but convenient approximation to provide a general idea of the situation? This often allows effort in other areas to proceed and, in addition, allows for a more effective concentration of refined calculation in the areas of most importance.

Are neutrons of all energies equally important to the behavior of the chain-reacting system, or are certain neutron energies of particular importance? In this latter case perhaps a treatment of a limited number of neutron energies will suffice and will help to reduce the mathematical complexity.

Is the majority of the reactor volume distant from boundaries by a mean free path or more? If so, certain valuable simplifications can be made in the form of the equations to be used.

Or, in the other extreme, is the reactor composed of a structure so fine that all detailed description can be neglected? It may then be possible to treat the entire medium as homogeneous if suitable care is exercised in formulating the equivalent homogeneous material.

Will the system behavior be sensitive to details of geometrical shape? If it will not, then simplified geometries may be used for calculation with great reduction in the effort because of a reduction in the number of dimensions to be handled or because of the occurrence of simpler mathematical functions.

Are important variations in composition of the system of a gradual nature? It may be entirely satisfactory to represent these variations by choosing several regions of uniform composition and connecting them with appropriate boundary conditions.

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Great reduction in over-all calculational effort can be achieved with little reduction in accuracy by judicious selection of approximate methods. However, cases will occur where very few approximations can be made with accuracy, and the system will, in the end, have to be studied in an experimental mock-up or critical experiment in order to establish sufficient confidence in the prediction of final system performance.

1 NEUTRON TRANSPORT

The study of the physical processes which may occur to a neutron in a system composed of scattering and absorbing materials is called neutron transport. In neutron transport no attempt is made to obtain a chain reaction or, for that matter, to treat any other processes except those which cause a change in the space or energy coordinates of the neutron. The combination of neutron transport with other physical phenomena to form a power-producing, chain-reacting system is treated in Art. 5.

1.1 The Boltzmann Equation

It is possible to derive an equation which rigorously describes the behavior of neutrons in any scattering or slowing-down material. This equation is based on the fundamental physical principle of neutron conservation within a closed system.

Consider the vector neutron flux

$$\phi(v, \Omega, \mathbf{r})$$

which is a vector whose magnitude ϕ is the number of neutrons of speed v which in one second cross a unit of area perpendicular to the vector direction Ω at the position given by the vector \mathbf{r} .

The equation for the conservation of neutrons is, then, composed of terms which represent the various possible sources and sinks in a system. The net number of neutrons leaving a unit volume per unit time in the speed interval dv and the direction interval $d\Omega$ is

$$\nabla \cdot \phi(v, \Omega, \mathbf{r}) dv d\Omega$$

The number of neutrons removed from this unit volume and this speed interval and direction interval by all forms of interaction with other nuclei, including slowing-down, is

$$N\sigma(v)\phi(v, \Omega, \mathbf{r}) dv d\Omega$$

where N = atomic density of the medium, atoms/cm³

$\sigma(v)$ = cross section for all removal interactions, cm²/atom (For an isotropic medium with no crystal effects this is independent of Ω)

The number of neutrons introduced into the unit volume and into this speed and direction interval by slowing-down collisions involving a change in speed v and/or a change in direction Ω will be equal to the number of these collisions integrated over all possible original speeds and directions. This source is, then,

$$dv d\Omega \iint_{v' \Omega'} \phi(v', \Omega', \mathbf{r}) N \sigma_s(v, v', \Omega, \Omega') dv' d\Omega'$$

where the primed quantities represent the speed and direction of the neutron before the interaction and where the quantity $\sigma_s(v, v', \Omega, \Omega')$ is a form of differential cross section and is a function of the initial and final speeds and of the difference between the initial and final directions, although not of either direction separately in an isotropic medium.

Finally, the source of neutrons introduced into the unit volume and into this velocity interval by external means is taken to be

$$S(v, \Omega, \mathbf{r}) dv d\Omega$$

This term can be used to represent neutrons born in fission or introduced by some external means.

Clearly if the rate of change of neutrons within the unit volume and this velocity (speed and direction) interval is to be zero (that is, the system is in steady state), then the number of neutrons entering the unit volume and this velocity interval must equal the number of neutrons leaving the same volume and velocity interval. Equating these two quantities yields

$$\nabla \cdot \phi(v, \Omega, r) + N\sigma(v)\phi(v, \Omega, r) = S(v, \Omega, r) + \iint_{v', \Omega'} \phi(v', \Omega', r) N\sigma_s(v, v', \Omega, \Omega') dv' d\Omega'$$

This is the steady-state Boltzmann equation and is the basic equation governing the behavior of neutrons in a system containing arbitrary scattering and absorbing materials. To be completely general the atomic density and the cross sections must be considered to be functions of position, but it will suffice to treat the case for a uniform medium.

If all the cross sections and sources involved were known experimentally, and if the equation could be solved, a rigorous description of the neutron flux would result. Unfortunately, the Boltzmann equation is an integrodifferential equation involving source and cross section functions of an arbitrary nature. As a consequence, it cannot be solved exactly for any but a few highly restricted cases. For most engineering purposes, approximations to this equation or approximate reformulations of it are of most use. To obtain equations of manageable difficulty it is usually necessary to make approximations as to both the space-wise and the energy-wise behavior of the neutrons. Below are treated briefly a number of such approximations to the Boltzmann equation which have been of use in practical problems.

1.2 The Space-independent Integral Equation

For cases where the neutron distribution is space independent or where the flux is slowly varying, it is possible to make a reduction in the complexity of the Boltzmann equation. For a region of constant flux the divergence term is zero. Then integrating over the various directions of the velocity vector yields

$$N\sigma(v)\phi(v) = S(v) + \int_{v'} \left[\iint_{\Omega'} \phi(v', \Omega') N\sigma_s(v, v', \Omega, \Omega') d\Omega d\Omega' \right] dv'$$

This space-independent Boltzmann equation is then rearranged into an equation for the collision density ψ . The collision density is equal to the neutron flux times the total probability of a collision (the cross section). Consequently, the space-independent equation can be made to take the form of an integral equation (in the energy variable E instead of the speed v)

$$\psi(E) = \int_{E'} \frac{\sigma_s(E')}{\sigma_t(E')} g(E, E') \psi(E') dE' + S(E)$$

where the kernel $g(E, E')$ is now the probability that a neutron of energy E' will be slowed down to energy E as the result of a scattering collision which it undergoes with probability $\sigma_s(E')$.

Solutions of this equation depend upon the choice of the kernel $g(E, E')$. Because this kernel will have energy cutoffs for most elements (that is, a neutron can lose only a fixed maximum fraction of its incident energy upon collision with an atom of mass greater than the neutron mass) both limits of the integral will become functions of the energy E . This situation does not lead to a form of the equation convenient for solution. Thus, the variable limit on the integral is usually avoided by using one of a number of synthetic kernels. These kernels do not give an exact description of the slowing-down process but are approximate functions which do not differ greatly from the correct function and which will yield a convenient reduction of the integral. A discussion of these synthetic kernels is included in several references.^{1,†}

† Superscript numbers refer to References at end of subsection.

1.3 The Hydrogen Equation

For the case of hydrogen a neutron may lose all its energy in one collision. The slowing-down kernel may be shown to have the form

$$g(E, E') = \frac{1}{E'} \quad \text{if } E \leq E'$$

$$= 0 \quad \text{if } E > E'$$

and only the lower limit on the integral is a function of the energy E . Consequently, the integral equation reduces directly to the differential equation

$$\frac{d\psi(E)}{dE} + \frac{\sigma_s(E)}{\sigma_t(E)} \frac{\psi(E)}{E} - \frac{dS(E)}{dE} = 0$$

which may be solved explicitly. The solution is

$$\psi(E) = N\sigma_t(E)\phi(E) = \frac{1}{E} \int_E^\infty \frac{\sigma_s(E')}{\sigma_t(E')} S(E') \exp \left[- \int_E^{E'} \frac{\sigma_a(E'')}{\sigma_t(E'')E''} dE'' \right] dE' + S(E)$$

1.4 The One-velocity Boltzmann Equation

Let us return now to the original Boltzmann equation (see Art. 1.1) and seek methods of solution which do not require the assumption of a space-independent neutron distribution. Quite obviously there will be many engineering cases where such an approximation will be useless. However, there are often cases where the energy dependence of the neutron flux can be neglected. In these cases the Boltzmann equation reduces immediately to

$$\nabla \cdot \phi(\mathbf{r}, \Omega) + N\sigma\phi(\mathbf{r}, \Omega) = \int_{\Omega'} N\sigma_s(\Omega, \Omega') \phi(\mathbf{r}, \Omega') d\Omega' + S(\mathbf{r}, \Omega)$$

This equation forms the basis for a great many engineering calculations and is of particular value in treating the distribution of thermal neutrons in a reactor.

The most common method for treating this equation is the spherical harmonics method which is discussed in Art. 2. The principal result of such treatment is the approximate reduction of this integrodifferential equation to a differential equation called the diffusion equation:

$$D \nabla^2 \phi(\mathbf{r}) - N\sigma_a \phi(\mathbf{r}) + S(\mathbf{r}) = 0$$

where D is the diffusion constant, a property of the medium. This equation can be solved explicitly for a number of cases of particular interest. It can also be shown that this equation yields solutions which are the asymptotic solutions of the one-velocity Boltzmann equation at points distant from boundaries and in low-absorption media. For this reason it forms the basis for many special methods discussed later on and has been used extensively for obtaining a first, reasonable estimate of the characteristics of many engineering systems.

A special form of the one-velocity Boltzmann equation can be obtained from consideration of the collision density ψ , which was discussed earlier in connection with the integral form of the space-independent equation. If ψ is the collision density and the kernel $K(|\mathbf{r} - \mathbf{r}'|)$ is defined as the probability that a neutron which starts at \mathbf{r}' will have its first collision at \mathbf{r} , then the number of neutrons per unit volume which will have their first collision at \mathbf{r} is given by

$$\psi_1(\mathbf{r}) = \int_{\mathbf{r}'} S(\mathbf{r}') K(|\mathbf{r} - \mathbf{r}'|) d\mathbf{r}'$$

where $S(\mathbf{r}')$ is an isotropic source which may vary with position and where the kernel is given by

$$K(|\mathbf{r} - \mathbf{r}'|) = \frac{\sigma_t}{4\pi|\mathbf{r} - \mathbf{r}'|^2} e^{-\sigma_t|\mathbf{r} - \mathbf{r}'|}$$

$K(|\mathbf{r} - \mathbf{r}'|)$ is the product of the inverse-square-law attenuation which is characteristic of a point source times the exponential probability that the neutron will arrive at the point \mathbf{r} , having traveled the path length $|\mathbf{r} - \mathbf{r}'|$ without having a collision, times the probability of its having a collision at \mathbf{r} . The latter is simply the total collision cross section at \mathbf{r} . If all scattering and absorption events are now regarded as absorptions, with $1 + f$ neutrons being emitted as the result of the absorption, then all collisions can be regarded as first collisions. By definition

$$1 + f(\mathbf{r}) = \frac{\sigma_s(\mathbf{r}) + \nu\sigma_f(\mathbf{r})}{\sigma_t(\mathbf{r})}$$

where ν is the number of neutrons emitted per fission. The equation for the collision density then becomes (where all collisions are regarded as first collisions)

$$\psi(\mathbf{r}) = \int_{\mathbf{r}'} K(|\mathbf{r} - \mathbf{r}'|) \{S(\mathbf{r}') + [1 + f(\mathbf{r}')]\psi(\mathbf{r}')\} d\mathbf{r}'$$

This is the integral formulation of the one-velocity Boltzmann equation. Note, however, that in the conversion of all collisions to first collisions the correlation between incident and scattered neutron directions has been lost. To this extent, this equation is less general than the integrodifferential formulation of the one-velocity Boltzmann equation.

Integration of the point kernel $K(|\mathbf{r} - \mathbf{r}'|)$ over various geometries results in the derivation of simpler forms of kernels for use in restricted geometries. The forms of the line, plane, and cylindrical and spherical shell kernels are given in Art. 2.25 of Sec. 6-2 and in Refs. 2.

1.5 The Fermi Age Equation

Let us examine now several techniques for obtaining approximate solutions to the Boltzmann equation where both space and energy variations in the neutron flux must be treated. Of these methods the simplest, and perhaps most direct, is the Fermi age technique. The age equation is based upon the continuous slowing-down model, which states that in each collision with a moderating atom a neutron loses an amount of energy characterized by the mean logarithmic energy decrement

$$\xi = 1 + \frac{(A - 1)^2}{2A} \ln \frac{A - 1}{A + 1}$$

and that for sufficiently heavy nuclei (where A is sufficiently large), the slowing down of the neutron can be approximated by a continuous history with a parameter ξ . This picture holds only very roughly for light elements where a neutron may lose a large amount of energy in one collision. For heavier elements the continuous slowing-down model is, however, very satisfactory. One important result of the continuous slowing-down model is the identification of the slowing-down density

$$q(\mathbf{r}, E) = \xi N \sigma_s \phi(\mathbf{r}, E)$$

which is the number of neutrons slowing down below a given energy E per unit time per unit volume.

The diffusion equation, including a slowing-down source, is written

$$D \nabla^2 \phi(\mathbf{r}, E) - N \sigma_a \phi(\mathbf{r}, E) + \frac{\partial q(\mathbf{r}, E)}{\partial E} + S(\mathbf{r}, E) = 0$$

Introducing the continuous slowing-down model and defining the Fermi age

$$\tau = \int_E^{E_0} \frac{D}{\xi N \sigma_s} \frac{dE'}{E'}$$

the age equation with absorption results:

$$\nabla^2 q(r, \tau) - \frac{N\sigma_a}{D} q(r, \tau) - \frac{\partial q(r, \tau)}{\partial \tau} + S'(r, \tau) = 0$$

If the resonance escape probability is defined as

$$p(E, E') = \exp\left(-\int_E^{E'} \frac{\sigma_a}{\xi\sigma_s} \frac{dE''}{E''}\right)$$

then the redefinition of the slowing-down density as

$$q = p(E, E')q'$$

gives immediately the result that

$$\nabla^2 q'(r, \tau) - \frac{\partial q'(r, \tau)}{\partial \tau} + \frac{1}{p(E, E')} S'(r, \tau) = 0$$

This shows that the solution with absorption, upon proper treatment of the external source term, is simply the solution of the Fermi age equation without absorption multiplied by the resonance escape probability. Finally, since many cases of interest have a source term only at high energy and are source-free at all lower energies, the equation reduces simply to the most familiar form of the age equation:

$$\nabla^2 q(r, \tau) - \frac{\partial q(r, \tau)}{\partial \tau} = 0$$

The similarity of this equation to that for the diffusion of heat should be noted. The quantity τ plays the role of time in the heat equation. Because of this it is called the "Fermi age" in the present application. It is, however, not a time variable but rather has the units of length squared. It can be identified with one sixth of the second moment of the neutron distribution and is thus one-sixth of the mean square distance traveled by the neutron in being slowed down to energy E .

1.6 The Selengut-Goertzel Equation

A very useful approximate solution to the Boltzmann equation has been given by Selengut and Goertzel.³ Recognizing the basic difference between slowing down in hydrogen and slowing down in all other materials, these authors have treated the processes separately. The Selengut-Goertzel equation, in its usual form, is an extension of the diffusion equation in which that part of the source which is due to slowing down in hydrogen is treated exactly while the remainder of the source is treated with the continuous slowing-down model. Any correlation between the deflection of the neutron and the degradation resulting from a collision with a hydrogen atom is ignored. The resulting equation has the form

$$D \nabla^2 \phi(r, E) - (N_{H\sigma_{sH}} + N\sigma_a) \phi(r, E) + S(r, E) + \frac{\partial q(r, E)}{\partial E} + \int_E^\infty \phi(r, E') N_{H\sigma_{sH}}(E') \frac{dE'}{E'} = 0$$

The Selengut-Goertzel equation may be solved numerically by incorporation into a multigroup formulation or by other numerical methods. It has proved to be reasonably convenient for a number of different kinds of problems. Note that the hydrogen scattering $N_{H\sigma_{sH}}$ appears added to the absorption $N\sigma_a$.

This method, while clearly approximate, nevertheless avoids treating hydrogen by continuous slowing-down theory and thus avoids the major approximation incorporated in the otherwise useful age theory. The Selengut-Goertzel method has met with surprising (and perhaps fortuitous) success in the description of processes of several types in water-moderated systems. More exact methods, which are much

more tedious, sometimes give results which agree less well with experiment than does the Selengut-Goertzel method. For this reason the Selengut-Goertzel method occupies a special position because of its unexpected accuracy.

1.7 The Multigroup Equations

Another very useful approach to the energy- and space-variation problem is obtained by approximating the continuous neutron energy spectrum by a finite number of energy groups. Within the i th group, which extends from E_{i-1} to E_i , neutrons are assumed to diffuse according to the one-velocity diffusion equation until they have suffered a number of collisions $(1/\xi) \ln (E_{i-1}/E_i)$, after which they move on to the next group. The group equation may be written in the form

$$D \nabla^2 \phi(r) - N \sigma_a \phi(r) + S(r) + \frac{\partial q(r)}{\partial E} = 0$$

Here q is the slowing-down density. The source term $S(r)$ includes neutrons introduced into the system from fission or by inelastic scattering. In the latter case, σ_a includes inelastic scattering out of the group in question. Integrating this equation over the energy width of the i th group yields the multigroup equation

$$D_i \nabla^2 \phi_i(r) - N \sigma_{a,i} \phi_i(r) + S_i(r) + q^I(r) - q^O(r) = 0$$

$q^O(r)$ and $q^I(r)$ are, respectively, q "out" and q "in," the slowing-down densities at the low-energy and high-energy limits of the group. The constants D_i and $N \sigma_{a,i}$ are now averages over the width of the group, and the source $S_i(r)$ is integrated over the group width.

For a small number of groups, usually two or three, the equation is formulated as given above. In particular, for two groups

$$\begin{aligned} D_1 \nabla^2 \phi_1(r) - N(\sigma_{a1} + \xi_1 \sigma_{s1}) \phi_1(r) + S_1(r) &= 0 \\ D_2 \nabla^2 \phi_2(r) - N \sigma_{a2} \phi_2(r) + \xi_1 N \sigma_{s1} \phi_1(r) &= 0 \end{aligned}$$

where the subscript 1 indicates the so-called "fast" neutron group and the subscript 2 indicates the "thermal" group.

However, for a large number of groups (over 40 have been used on occasion) the multigroup equation is usually written as an equation for the slowing-down density, using the continuous slowing-down model

$$\left(\frac{D_i}{\xi_i N \sigma_{s,i}} \right) \nabla^2 q_i(r) - \left(\frac{N \sigma_{a,i}}{\xi_i N \sigma_{s,i}} \right) q_i(r) + S_i(r) + q^I(r) - q^O(r) = 0$$

where the bracketed quantities are again averages over the group width and where the slowing-down density, $q_i(r)$, and the source are integrated over the lethargy width of the group. The lethargy variable is defined as

$$u = \ln \left[\frac{10 \text{ Mev}}{E(\text{Mev})} \right]$$

The reference energy need not be chosen to be 10 Mev; however, it has become customary to do so.

For the highest energy group (the lowest lethargy), $i = 1$, and if $q^I(r)$ and $S_1(r)$ are known, then the equation for this group may be solved for $q^O(r)$, provided that some relation among $q_1(r)$, $q^O(r)$, and $q^I(r)$ is established. This value of $q^O(r)$ for the first group is then identified as the value of $q^I(r)$ for the second group, and the process is continued down through the remaining groups until the lowest energy (highest lethargy) group is reached.

The relation between the q 's is usually assumed to be of the form

$$q_i = w_1 q^I + w_2 q^O$$

where w_1 and w_2 are constants to be chosen. A number of ways of choosing these constants is discussed in Ref. 4.

Care is required in choosing the cross sections for each medium in each group. The correct formulation for these constants can be obtained from the averaging process indicated above. For a very large number of groups the group constants are usually calculated from the cross-sectional curves directly. For two-group work, however, the constants needed are usually taken from experimental data where possible, using the relations

$$\begin{aligned} D_2 &= L^2 N \sigma_{a2} \\ D_1 &= \tau N (\sigma_{a1} + \xi_1 \sigma_{s1}) \end{aligned}$$

where L^2 is the thermal diffusion area and τ is the Fermi age of neutrons entering the thermal energy group.

Finally, it should be noted that the techniques of multigroup theory, while usually applied to the form of the diffusion equation discussed above, are not limited to this particular application and can be used with various other forms of the Boltzmann equation. In particular, the Selengut-Goertzel equation has been used, as has the form of the diffusion equation with a variable diffusion constant. In this latter case, the Laplacian term becomes

$$\text{div } D_i(\mathbf{r}) \text{ grad } \phi_i(\mathbf{r})$$

1.8 The Greuling or Diffusion Kernel Equation

An alternative formulation of the group picture stems from the integral form of the one-velocity Boltzmann equation. This formulation is most simply illustrated by considering a point, isotropic, unit source of neutrons at position \mathbf{r}_0 . If the medium is infinite and nonabsorbing, then the solution of the one-velocity integral equation for the slowing-down density is

$$q_1(\mathbf{r}) = K_1(\mathbf{r}, \mathbf{r}_0)$$

where K_1 is the point diffusion kernel and is defined as the probability that a neutron which is born at \mathbf{r}_0 will be removed from the group by an elastic scattering collision at \mathbf{r} . The point diffusion kernel is similar in its effect to the point transport kernel discussed earlier (see Art. 1.4) except for the change in definition of the units involved. The slowing-down density and the collision density ψ are related by

$$q = \frac{\xi \psi}{\Delta u}$$

If the energy interval, Δu , described by this equation for the slowing-down density is wide compared with the energy loss in one collision, so that the continuous slowing-down model applies, then the slowing-down density resulting from this first group may be used as a source for the next lower energy group of neutrons, for which the solution for the slowing-down density is immediately

$$q_2(\mathbf{r}) = \int_{\mathbf{r}'} K_2(\mathbf{r}, \mathbf{r}') K_1(\mathbf{r}', \mathbf{r}_0) d\mathbf{r}'$$

This process may be continued for any number of groups, the solution for the slowing-down density for the n th group being merely the convolution of the n slowing-down kernels which apply to the higher energy groups. In an absorbing medium a very similar result can be obtained. The solution for q_n as a convolution of kernels is valid for other geometric diffusion kernels provided that the material cross sections are independent of position in the medium.

This particular integral form of the group equations uses the continuous slowing-down picture. Consequently, the Fermi age equation can be shown to be the limiting case of this group formulation as the number of groups goes to infinity. Also, as in the case of the space-independent integral equation, various forms of the kernels can be used for different physical models in order to give alternative, approximate approaches to the description of the energy and space variations of the neutron flux.

2 THE SPHERICAL HARMONICS METHOD AND THE DIFFUSION EQUATION

The one-velocity Boltzmann equation forms the basis for many calculations in engineering practice. The most common technique for treating this equation is the spherical harmonics method. In this article a number of approximations based upon this method will be discussed with emphasis on their relative validities in physical problems.

2.1 The Spherical Harmonics Method

Basically the spherical harmonics technique consists of expanding the source and flux distributions in spherical harmonics of the cosine of the angle between the direction Ω and some principal coordinate direction. The source and flux distributions are then considered to be rotationally invariant with respect to this coordinate direction. If μ is the cosine of the angle between the direction Ω and the reference coordinate, then the flux is given by

$$\phi(r, \mu) = \sum_{m=0}^{\infty} \left(\frac{2m+1}{2} \right) \phi_m(r) P_m(\mu)$$

where

$$\phi_m(r) = \int_{-1}^1 \phi(r, \mu) P_m(\mu) d\mu$$

The source term is treated in a similar fashion. The quantity $P_m(\mu)$ is the Legendre polynomial of order m .

The cross section $\sigma_s(\Omega, \Omega')$ is also expanded in harmonics of the cosine of the angle between the incident and the scattered particles. This series may then be converted into an expansion in the cosine of the direction angle μ by means of the addition theorem for spherical harmonics. The scattering term in the one-velocity Boltzmann equation then becomes

$$\frac{1}{2} \sum_{m=0}^{\infty} (2m+1) N \sigma_{sm} P_m(\mu) \int_{-1}^1 P_m(\mu') \phi(r, \mu') d\mu'$$

where

$$\sigma_{sm} = \int_{-1}^1 \sigma_s(\Omega \cdot \Omega') P_m(\Omega \cdot \Omega') d(\Omega \cdot \Omega')$$

The spherical harmonics method then assumes that all terms containing harmonics higher than $P_n(\mu)$ contribute negligibly to the result. This approximation to the solution is known as the P_n approximation. The resulting approximate equation is multiplied in turn by each spherical harmonic $P_n(\mu)$ and is integrated over $-1 \leq \mu \leq 1$. Because of the orthogonality of the spherical harmonics, this results in a set of $n+1$ first-order differential equations.

2.2 The Diffusion Equation

In the P_1 approximation it is assumed that, in the expansions, contributions from terms beyond the first two are negligible. It can be shown that this assumption implies that the cross section angular distribution can be expressed in the form

$$\sigma_s(\Omega, \Omega') = A + B \cos(\Omega \cdot \Omega')$$

and that the medium involved is only weakly absorbing. Also, because the higher harmonics express transient effects near boundaries, the P_1 approximation assumes that the region in question is several mean free paths from any boundary.

The two resulting equations can be combined into a single equation for the isotropic component of the flux.

$$D \nabla^2 \phi(\mathbf{r}) - \Sigma_a \phi(\mathbf{r}) + S(\mathbf{r}) = 0$$

where

$$D = \frac{1}{3\Sigma_s(1 - \bar{\mu})}$$

where $\bar{\mu}$ is the average value of $\cos(\Omega \cdot \Omega')$. The source here has been assumed to be isotropic.

This equation is the so-called diffusion equation and is of particular use in reactor physics because it can be solved analytically for many geometries. It provides satisfactory accuracy as long as the conditions for validity of the approximations mentioned above are met. In particular this equation finds use in describing the distribution of flux of any monoenergetic neutron group within a large medium such as a reactor core. The diffusion equation is also often used as the basis of the derivation of the flux disadvantage factors within a unit cell of a heterogeneous reactor lattice. The validity of this last application is, however, open to question, since most lattice cells contain strongly absorbing material and because the regions involved are not large compared with a mean free path. Because of its convenience, however, the diffusion equation is very often used to derive a first approximation to these flux disadvantage factors.

2.3 The P_2 Approximation

It can be shown that the P_2 approximation yields the same differential equation as does the P_1 approximation. The only difference is that certain correction terms occur to the constants in the equation. In particular, the diffusion equation resulting from the P_2 approximation is

$$D \nabla^2 \phi(\mathbf{r}) - \Sigma_a \phi(\mathbf{r}) + S(\mathbf{r}) = 0$$

with

$$D = \frac{1}{3\Sigma_s(1 - \bar{\mu})} \left[1 + \frac{4}{5} \frac{\Sigma_a}{\Sigma_s(2 - \frac{3}{2}\bar{\mu}^2)} \right]$$

While this equation yields improved values of the constants, the generality of its application is not greatly improved and the remarks in the previous section apply. Again, the approximation is valid only for the case of weak absorption where

$$\frac{\Sigma_a}{\Sigma_s} \ll 1$$

2.4 The P_3 Approximation

If it is assumed that all terms beyond the fourth contribute negligibly to the result, a distinct improvement in the generality of the solution is obtained. In particular the equation which results for the flux is now of fourth order and has solutions containing terms which affect the flux particularly in the region of boundaries.

Experience has shown that the P_3 approximation provides a much more accurate description of the neutron flux within a reactor lattice cell. Generally the P_3 flux distribution is entirely satisfactory within the fuel element and is fairly good in the surrounding moderator. Comparison with higher harmonics (P_5 , P_7 , etc.) has shown that in most cases of interest the P_3 approximation is satisfactory and provides most of the needed improvement over diffusion theory.

The P_3 approximation involves the solution of large algebraic systems and is usually carried out on digital computing machines.

It should be recognized that although the P_3 approximation effectively provides a correct description, for most physical cases, of the neutron flux distribution for a one-velocity group of neutrons, many cases of interest are not correctly described by such a one-velocity model. Because of its complexity the P_3 approximation has not been widely applied to two-or-more-velocity approximations although some work has been done on a two-group approach.⁵

3 OTHER TECHNIQUES FOR TRANSPORT PROBLEMS

In this section several techniques of a somewhat unconventional nature will be considered. Historically, the solutions to most reactor problems in neutron transport have been obtained with the methods which have been described in the previous sections. Recently, however, two suggestions have been made concerning other approaches to the above equations. These two ideas are the hemispherical harmonics method and the S_n method. In addition, an entirely different technique of great general applicability has been developed. The Monte Carlo method, as it is called, may become important in the future for the solution of complex problems of all kinds if the abilities of computing machinery continue to increase at their present rapid rate.

3.1 Hemispherical Harmonics

The method of spherical harmonics, which has been discussed in some detail earlier, has proved of great use in many applications. However, the rate of convergence of the expansion in spherical harmonics is not known, and the success or failure of the method has been determined largely by numerical computations and by comparison with experiment. The complexity of solving the higher approximations in spherical harmonics has limited the use of the method, in most cases, to the P_3 approximation. It would be desirable to have a method which would retain the advantages of the spherical harmonics method but which would converge more rapidly, thus yielding equivalent accuracy with less algebraic complexity.

It has been suggested that in most cases of interest the majority of neutrons flow in the directions of the principal coordinate axes. That is, the boundaries between regions are often surfaces of constant radius r or displacement x . Consequently, the gradient of flux will often be directed perpendicularly across these boundaries. The spherical harmonics method, however, expands the flux in a series of harmonics containing some components which are largely in a direction parallel to the boundaries. Improved convergence might be obtained if an expansion were used which eliminated components of importance which were parallel to the boundary. This is what the hemispherical harmonics method attempts to do.

Instead of a single expansion in μ , the cosine of the angle between the velocity vector and the principal coordinate direction, the flux is expanded in two different ways. That part of the flux with μ between -1 and 0 is expanded in harmonics of the variable $(2\mu + 1)$, while that part of the flux with μ between 0 and 1 is expanded in harmonics of the variable $(2\mu - 1)$. In this way two complete expansions are derived, the sum of which represents the neutron flux as a function of angle at the space point in question. As with the spherical harmonics method, these expansions are truncated arbitrarily after a given number of terms to achieve an approximation. For consistency, the two expansions are cut off after the same number of terms. Because the functions involved are still Legendre polynomials, the approximations are called $P_{n,n}$ for the case where each expansion is cut off after $n + 1$ terms.

Although not much work using this approximation has appeared in print, it is expected to have considerably improved convergence characteristics, with a $P_{2,2}$ approximation yielding accuracy comparable to a P_3 approximation in the spherical harmonics method, but with considerably reduced algebraic complexity. Unfortunately, however, the hemispherical harmonic equations are not analytically solvable in cylindrical geometry. Because of the great interest in this type of configuration, however, numerical solution of the $P_{n,n}$ equations will probably be attempted for this case in the near future.

An additional improvement is to be expected from the hemispherical harmonics method in the treatment of black boundaries. Because the hemispherical harmonics do not contain flux components parallel to most boundaries, a more correct description is to be expected in the vicinity of absorbing boundaries.

The method of hemispherical harmonics is sometimes referred to as the method of Yvon.⁶

3.2 The S_n Method

The S_n method of Carlson⁷ is still another approach to the solution of the steady-state Boltzmann equation. It is usually applied to the one-velocity equations which are used in multigroup theory. However, it does not require the diffusion approximation but is applied to the case where, for each energy group, the one-velocity Boltzmann equation holds with but two restrictions. The source term is taken to be isotropic, and energy degradation is taken to be independent of the deflection angle of the scattered particle.

The S_n method then consists of approximating the flux as a function of direction, $\phi(\mathbf{r}, \mu)$, by a series of straight-line segments

$$\phi(\mathbf{r}, \mu) = \frac{\mu - \mu_{j-1}}{\mu_j - \mu_{j-1}} \phi(\mathbf{r}, \mu_j) + \frac{\mu_j - \mu}{\mu_j - \mu_{j-1}} \phi(\mathbf{r}, \mu_{j-1})$$

where

$$j = 0, 1, 2, \dots, n$$

The order of the approximation is characterized by the number of line segments n which is used. This approximation is then used to reduce the one-velocity Boltzmann equation to a set of n equations in the $n + 1$ variables $\phi(\mathbf{r}, \mu_j)$. An additional equation is obtained by setting $\mu = -1$ directly in the one-velocity Boltzmann equation.

The resulting set of equations is then solved numerically for the fluxes, an initial guess for the sources having been made. The equations for the various energy groups are solved in order of decreasing energy, and a new approximation to the fission source is obtained. The process is continued until an acceptably small change is obtained between two successive source calculations.

The S_n method has been applied to time-dependent problems as well as to the steady-state problem outlined above. For the stationary case the anisotropic problem (which includes the correlation between scattering angle and energy degradation) has been solved for spherical geometry.

The chief value of the S_n method would appear to be its ease of adaptation to calculation on digital machines. It has been applied to a large number of varied problems, chiefly at the Los Alamos Scientific Laboratory.

One over-all comment is applicable to the S_n method. Although it is usual to choose the n intervals in μ as of equal size, it is possible to choose unequal intervals. For a given value of n the most efficient choice of intervals would appear to be that given by the Gaussian integration formula,⁸ and it can be shown that this choice of Gaussian intervals will yield values for the isotropic flux exactly equal in accuracy to those obtained with the spherical harmonics approximation with the same n . (The angular distributions will, however, not be the same as given by the spherical harmonics method.) Therefore, it would appear that the S_n method would be less efficient than the corresponding spherical harmonics approximation. However, it may have advantages in ease of application which would outweigh the reduced efficiency in convergence of the approximation.

Sykes⁹ has reported a method using a double Gaussian integration. Such a technique would appear to be a numerical equivalent of the hemispherical harmonics method of Yvon.

3.3 The Monte Carlo Method

A new technique, which is of a different nature from any discussed so far, is gaining importance for use in transport problems as more powerful calculating machines become available. This technique applies the statistical methods of random sampling to those physical and mathematical problems to which a probability analysis is applicable either directly or as an analogue. Because of the random-sampling approach, in which random numbers are actually used, the technique has acquired the name of the "Monte Carlo" method.

The history of a neutron in a chain reactor may be represented accurately as a sequence of statistical events. A cross section is, by definition, the probability that a neutron will suffer a collision in traversing a given distance of material. Thus, a neutron may be introduced into a system at some random point with an energy chosen at random from the distribution of fission neutrons and with random orientation. Successive random selections are then made to determine, with appropriate distributions of cross sections and other material constants, when and where the neutron is scattered or slowed down or absorbed. The process is continued until the neutron is captured or escapes from the system. The average characteristics of the system, such as neutron diffusion, aging, and attenuation, can then be obtained from a statistical average of many such histories.

In general, the Monte Carlo method may be regarded as a technique for evaluating integrals or for solving integral or differential equations. Its chief virtue becomes evident for problems involving many dimensions in that the difficulty of solution increases roughly with the number of dimensions rather than with the n th power, where n is the number of dimensions, which is the case for a numerical technique such as Simpson's rule. In practice, it turns out that Monte Carlo is superior to other numerical approaches if the dimensionality of the problem exceeds 4 or 5.

With the Monte Carlo method, in principle, if the fundamental probabilities of the events are known (cross sections, etc.), any problem can be solved to any desired degree of accuracy. However, very time-consuming sampling and statistical calculations are often required. Consequently, Monte Carlo has been applied primarily to problems which have been prohibitively difficult to handle by other methods. In even these cases it may still be necessary to restrict the Monte Carlo analysis to certain aspects of the problem or to perform supplementary approximate calculations or experiments on aspects amenable to simpler treatment.

It is illustrative to consider two problems to which the Monte Carlo method has been applied. The very different requirements of these problems are indicated by the following formula, which is derivable from the binomial probability distribution:

$$N = \frac{1 - P}{\epsilon^2 P}$$

Here N is the number of case histories necessary to calculate the probability P of a process to within the relative error ϵ . The assumption is made that ordinary sampling procedures are used.

Consider now a problem in reactor shielding, where it is desired to know the probability of a given radiation penetrating a shield of some arbitrary composition and shape. For cases of engineering interest, P may be of the order of 10^{-10} , so that N is prohibitively large even though a 100 per cent error may be tolerable. Physically, the difficulty is that an attempt is being made to get statistical accuracy in the number of radiations penetrating the shield. But a very large number of radiations must be followed in order to find just one which succeeds in getting all the way through. The situation is made still more difficult if the composition or geometry of the shield is complex, since individual particle histories then become tedious to calculate. A considerable amount of work has gone into obtaining improved mathematical techniques for reducing the number N of histories which are required.¹⁰ Basically, these improvements deal with sampling methods which sample more efficiently the particles having a larger probability of penetrating the shield. The improvements also attempt to obtain more information from a given history or to perform a suitable transformation of the problem into a related one in which the probability of success P is considerably larger.

As a second example, consider the calculation of the resonance escape probability in a heterogeneous lattice moderated with light water. Here each history is short, since only a small number of collisions are required to carry a neutron completely through the resonance region. Furthermore, the probability of success P is about 0.9, so that 10^3 to 10^4 histories will determine the resonance escape probability to an

accuracy of 1 per cent. This problem has been carried out successfully using ordinary random sampling.¹¹ It should be noted that although this is a case peculiarly suited to Monte Carlo calculation, several hours of Univac time are required to complete a single computation of resonance escape.

One further comment on the Monte Carlo method is appropriate. Suppose that a calculation is to be performed of the reactivity of a nuclear reactor. By random-sampling techniques it is, in principle, possible to determine the average number of neutrons produced by each neutron introduced into the system (that is, the reactivity). But the spatial and energy distributions of the neutrons will not be determined accurately by this calculation. A very much larger number of histories would have to be followed in order to determine, to satisfactory accuracy, the flux of neutrons of each energy at each point in the reactor. On the other hand, an analytical or conventional numerical solution of the Boltzmann equation will give not only the over-all characteristics of the system but the distributions of the neutron flux. This is because, in the analytical case, the over-all characteristics are computed from the solutions for the fluxes. From this comparison it may be seen that the Monte Carlo method is useful primarily where the desired results are a limited number of specific system characteristics rather than the complete solutions for the system.

The present time-consuming Monte Carlo method will, in the future, become practical as better computing machines develop and as improved sampling techniques provide greater calculational efficiency. With the rising cost of experiments and with the inevitable increase in the complexity of engineering nuclear systems, the Monte Carlo method can be expected to be brought increasingly into use. This will be true even though the first wave of enthusiasm for the method is now over and the practical difficulties are clearly realized.

4 THE TREATMENT OF BOUNDARIES IN NEUTRON TRANSPORT

Once solutions have been obtained for a specific material by any of the analytical methods discussed above, it is necessary to be able to join solutions for different media and to include geometrical restrictions on the size of the system. Although very simple methods can be used to achieve these definitions of the geometrical problem, there are more subtle techniques which can materially improve the accuracy of a calculation.

4.1 Simple Boundary Conditions

The fundamental condition for matching solutions across a boundary between regions is that of physical continuity. In particular, for the Boltzmann equation this condition may be stated that the flux, $\phi(v, \Omega, \mathbf{r})$, must be continuous across a boundary for all Ω and v . For the more practical reductions of the Boltzmann equation, this condition extends directly. Consider, as an illustration, the spherical harmonics equations. The condition of continuity requires that the components of the flux, into which the total flux has been expanded, shall each be continuous across a boundary. In the P_1 and P_3 approximations, the one-velocity Boltzmann equation reduces to two first-order differential equations for the flux components $\phi_0(\mathbf{r})$ and $\phi_1(\mathbf{r})$, where

$$\phi_n(\mathbf{r}) = \int_{-1}^1 \phi(\mathbf{r}, \mu) P_n(\mu) d\mu$$

The $\phi_0(\mathbf{r})$ component is merely the integrated isotropic flux, since $P_0(\mu)$ is a constant, independent of μ . Similarly, since $P_1(\mu)$ is equal to μ , the $\phi_1(\mathbf{r})$ component of the flux is directly interpretable as the net neutron current in the \mathbf{r} direction. Consequently, the boundary conditions for the spherical harmonics method in the diffusion approximation amount to requiring continuity of flux and neutron current across a boundary. In practice, it is sufficient for criticality calculations to reduce these two conditions to requiring that the diffusion constant times the logarithmic derivative of

the flux

$$D \frac{1}{\phi} \nabla \phi$$

be continuous across an interface.

Of course, the condition of continuity of flux and current (or of higher components of the flux in other approximations than the diffusion approximation) assumes that the interface is passive. In the event of an active interface, which may be either an absorber of neutrons or a source, an obvious conservation relation replaces the above continuity conditions.

4.2 Extrapolated End Points

Special treatment is required for the case of a black boundary or its equivalent, a boundary with an infinite region of vacuum. The former case may arise in the treatment of strongly absorbing media (such as control rods), and the latter may arise during the treatment of bare reactors where the active lattice has an outer boundary in common with a region assumed to be of infinite extent and which has the properties of a vacuum. The obvious boundary condition which may be imposed is that the flux shall go to zero at the boundary of such media. However, in the adjoining active region the flux does not physically go to zero at the boundary, since there will be a streaming of neutrons toward the infinite sink represented by the black or vacuum region. The solution to this seeming paradox has been obtained, in practice, from an examination of the desired quantities in the finished calculation and takes the form of the so-called extrapolated end point.

What is desired in calculations of criticality and in most flux distributions is a correct treatment of flux at relatively large distances from a boundary. In such regions, the transient terms in the solution to the equations will have disappeared and only the asymptotic solutions will remain. It would be desirable, thus, to require that the boundary condition imposed produce a correct asymptotic flux distribution.

In the commonly used diffusion approximation, the solution to the flux equation contains a limited number of transient terms and gives a physically correct description only at points far from boundaries. Put another way, the diffusion approximation can be shown to be the asymptotic solution of the transport equation in weakly absorbing media and away from boundaries. Consequently, it is desired to achieve a boundary condition such that the diffusion-theory solution will approach the correct transport solution at interior points in the region. Extrapolation of the correct asymptotic or diffusion solutions through the boundary and into the adjoining medium shows that if the flux is required to go to zero at this extrapolated boundary, then a correct asymptotic description will be obtained. This treatment, however, will not yield a correct description of the neutron flux in the vicinity of the boundary.

If the physically meaningful boundary condition is imposed that the current returning from the black or vacuum region should be zero, then solutions may be obtained which yield quantitative estimates of the size of the extrapolation distance which is to be used with a zero-flux boundary condition. A linear extrapolation of the diffusion solutions shows that, for a plane boundary, the extrapolation distance is $\frac{2}{3}\lambda_{tr}$. However, consideration of more accurate approximations to the Boltzmann equation and its solutions shows that a more correct extrapolation distance for the asymptotic solution is $0.71\lambda_{tr}$. The quantity λ_{tr} is the transport mean free path of the active medium and is equal to three times the diffusion constant D . Thus for a bare reactor, the flux is required to go to zero at an extrapolated boundary which is $0.71\lambda_{tr}$ into the adjoining vacuum medium.

The above values for the extrapolation distance apply for a plane boundary and for the case of small capture and isotropic scattering. Similar results for plane boundaries, with other conditions of capture or scattering, are given in Refs. 12. For most cases, however, the value of $0.71\lambda_{tr}$ is adequate.

For curved boundaries the extrapolation distance varies as a function of the radius of curvature. For a black sphere, the extrapolation distance increases monotonically to $\frac{4}{3}\lambda_{tr}$, in diffusion theory, as the radius goes to zero. This is physically because a

restricted black region (such as a sphere) depresses the flux in the adjoining medium less than a boundary which is very large (such as a plane).

The use of diffusion theory with the appropriate extrapolation distances for the boundaries involved will give rigorously correct values for the flux a few mean free paths from the boundary. For a very large class of cases, this treatment will yield satisfactory accuracy.

4.3 Serber-Wilson Boundary

Although the Serber-Wilson method is a technique for performing criticality calculations, it is included in this section because it is primarily a special case of the boundary conditions to be applied to transport problems. The Serber-Wilson method has been applied to the calculation of small, reflected reactors with considerable success.¹³ Although this has been its major use, it has also been applied to other problems, notably the calculation of the thermal flux distribution in a unit lattice cell of a heterogeneous reactor.¹⁴

The Serber-Wilson method is primarily for one-velocity problems and deals with the asymptotic solutions to the transport equation. The asymptotic forms of the solutions are assumed in each of the media concerned, and special boundary conditions are used to assemble these individual solutions into a single description of the physical system. Because conservation of neutrons is implied by the continuity of net current across a boundary, this condition is retained from the usual diffusion-theory approach. However, for the simplest case of a spherical reactor with an infinite reflector, the usual continuity of flux is not required, since, with a true solution to the transport equations, the asymptotic or diffusion solutions are not continuous at the boundary. For this simple example, only one additional equation is needed to replace the discarded flux continuity condition. The so-called "Serber condition" which is used requires that the one-velocity Boltzmann equation be satisfied exactly at the center of symmetry of the system, which is the origin in this case. This condition then allows the system to be completely described.

For this simple case, with a single infinite or finite reflector, numerical results are easy to obtain and are given in Ref. 13 in convenient form for the solution of problems.

For systems involving more than one intermediate boundary it is necessary to obtain such an additional equation for each boundary. This is accomplished by satisfying the Serber condition "in detail." In the simple spherical case, satisfaction of the Serber condition in detail results in the so-called "Wilson conditions," which require that the flux in the one-velocity Boltzmann equation $\phi(r, \mu)$ be continuous at all boundaries for the value $\mu = -1$. These radially inward fluxes are, however, to be computed as if the two adjoining media were infinite in extent.

Although the Serber-Wilson method has been useful for calculation of reflected systems, it has not generally been extended to the many-velocity case. Consequently, it can be helpful only where the nature of the system makes a one-velocity approach a reasonable approximation to reality.

4.4 Albedo

The albedo is a concept which has been of use for those systems where one-velocity diffusion theory is adequate. It is simply defined as the ratio of the current density out of the medium in question to the current density into the medium in question. The albedo is thus the fraction of neutrons which are scattered back across the boundary after any number of collisions and is, to this extent, a "reflection" coefficient. For the diffusion-theory case, the albedo can be shown to be a property of the reflecting medium only and, for this reason, allows certain simplifications to be made in the treatment of boundaries between a reactor core and a reflector. Inserting the definitions of neutron current into the definition of the albedo gives immediately

$$\text{Albedo} = \beta = \frac{1 + (2D/\phi)(d\phi/dx)}{1 - (2D/\phi)(d\phi/dx)}$$

Because of the required continuity of flux and current in diffusion theory, the quantities ϕ and $D d\phi/dx$ may be chosen for either medium. For a finite slab reflector the solutions of the diffusion equation can be inserted directly to give

$$\beta = \frac{1 - 2\kappa D \coth \kappa a}{1 + 2\kappa D \coth \kappa a}$$

where a is the slab thickness, including the extrapolation distance, and κ is the reciprocal of the diffusion length. For an infinite slab reflector this becomes

$$\beta = \frac{1 - 2\kappa D}{1 + 2\kappa D}$$

It will be noted that the albedo of a finite slab is less than that for an infinite slab because of the loss of neutrons by escape from the outer boundary. When the reflector is more than about two diffusion lengths thick, a finite reflector is essentially indistinguishable from an infinite reflector.

For spherical boundaries the expression for the albedo will be different, with the albedo of a medium being less when it surrounds a spherical source. This is because neutrons which get into the reflector an appreciable distance have a smaller solid angle into which they can be scattered and still return to the source region. Thus the albedo will depend both on the properties of the medium and on its geometrical shape. For other shapes see Art. 2.3 of Sec. 6-2.

A good reflecting medium will have a diffusion coefficient small compared with its diffusion length, as may be seen from the above expressions. The albedo is 90 per cent or over for infinite slabs of good reflecting materials such as heavy water or graphite. Because of its absorption, an infinite, light-water slab has an albedo of only 0.821 for thermal neutrons.

The albedo can be used to replace other forms of boundary conditions. For a plane boundary

$$\frac{1}{\phi} \frac{d\phi}{dx} = -\frac{1}{2D} \left[\frac{1 - \beta}{1 + \beta} \right]$$

and the extrapolation distance at the boundary is

$$d = 0.71\lambda_{tr} \left[\frac{1 + \beta}{1 - \beta} \right]$$

For the case of a nonscattering absorber or of vacuum, the albedo is zero and this equation reduces to the extrapolated end point treated earlier.

While it is a conceptually pleasing quantity, the albedo is limited in its reactor applications to cases where the one-velocity diffusion picture is adequate. Because of this, it has not found wide application to the more recent reactor systems, particularly those containing light water where important boundary effects like the rise in thermal flux at a core-reflector interface can be described only on a multiveLOCITY basis. The albedo has, however, been useful in shielding work.¹⁵

5 REACTOR EQUATIONS

Many problems in reactor engineering may be solved as transport theory problems directly. That is, they take no cognizance of the multiplying nature of the system. However, it is necessary in performing criticality calculations to include this important characteristic. The modification and extension of the methods of transport theory to include these problems will be treated in this section, along with a valuable tool for predicting the effects of small disturbances on a chain-reacting system.

5.1 The Reactor Equations and Their Relation to Transport Theory

The methods of transport theory have always included a provision for an arbitrary source of neutrons. In the reactor equations, this source is assumed to come from

neutrons born in fission. External sources are normally not included in critical systems, although they are often used during the approach to critical. In this section, however, only systems which are critical will be discussed, so that no provision for an external neutron source will be made.

In any reactor, the number of neutrons introduced into the system is proportional to the number of fissions, which, in turn, is proportional to the flux of neutrons which produce fission. For a uniform thermal reactor, then, the source of neutrons will be proportional to the thermal flux at each point.

Consider the simple case of a one-group diffusion equation. The source of thermal neutrons will be proportional to the flux, the factor of proportionality being the absorption cross section times the number of neutrons produced for each one absorbed at the point in question, or k_∞ . The one-group reactor diffusion equation is then

$$D \nabla^2 \phi(\mathbf{r}) + N \sigma_a (k_\infty - 1) \phi(\mathbf{r}) = 0$$

As has been discussed earlier, however, the one-group picture does not provide an adequate description for most engineering reactors. In a two-group picture, the source of fast neutrons is proportional to the thermal-neutron flux, the constant of proportionality being equal to $(k_\infty/p)N\sigma_{a2}$. Here p is the resonance escape probability. Assuming that resonance capture occurs between the fast and slow groups, the source of thermal neutrons is proportional to the fast flux with a constant of proportionality of $pN\xi_{1\sigma_{s1}}$. The quantity $N\xi_{1\sigma_{s1}}$ is defined as the macroscopic cross section for the removal of a fast neutron from the fast group by slowing down into the thermal group. Hence, the two-group reactor equations become

$$\begin{aligned} D_1 \nabla^2 \phi_1(\mathbf{r}) - N(\sigma_{a1} + \xi_{1\sigma_{s1}})\phi_1(\mathbf{r}) + \frac{k_\infty}{p} N \sigma_{a2} \phi_2(\mathbf{r}) &= 0 \\ D_2 \nabla^2 \phi_2(\mathbf{r}) - N \sigma_{a2} \phi_2(\mathbf{r}) + p N \xi_{1\sigma_{s1}} \phi_1(\mathbf{r}) &= 0 \end{aligned}$$

The one-group equation and the homogeneous part of the two-group equations may be written in the form of a wave equation:

$$\nabla^2 \phi(r) + B^2 \phi(r) = 0$$

Solutions of this equation for various geometries are well known.¹⁶ Use of this homogeneous solution in the two-group equations leads directly to

$$\begin{aligned} -[D_1 B^2 + N(\sigma_{a1} + \xi_{1\sigma_{s1}})]\phi_1 + \frac{k_\infty}{p} N \sigma_{a2} \phi_2 &= 0 \\ p N \xi_{1\sigma_{s1}} \phi_1 - (D_2 B^2 + N \sigma_{a2}) \phi_2 &= 0 \end{aligned}$$

where the constant B^2 is taken to be the same for both groups. By Cramer's rule, the condition for a nontrivial solution leads immediately to the two-group critical equation (for small absorption in the fast group)

$$k_\infty = (1 + \tau B^2)(1 + L^2 B^2)$$

The corresponding one-group equation is

$$k_\infty = 1 + M^2 B^2$$

In both of the above critical equations the following relations have been used:

$$\tau = \frac{D_1}{N \xi_{1\sigma_{s1}}} \quad L^2 = \frac{D_2}{N \sigma_{a2}} \quad M^2 = L^2 + \tau$$

Obviously, almost any transport theory approach can be used to produce solutions to the flux equations and to give a criticality equation. The approach can be made either through the integrodifferential formulations or through the integral equation formulations of the Boltzmann equation. One further form of the critical equation is worth noting here. On the basis of age theory (or from the integral equations, using Fermi age or Gaussian kernels) the rigorous critical equation can be shown to be (no

"group" approximations having been made)

$$k_{\infty} e^{-B^2\tau} = 1 + L^2 B^2$$

For a large reactor where the buckling B^2 is small, all the above equations can be reduced to the form of the one-group equation

$$k_{\infty} = 1 + M^2 B^2$$

In general, the remarks concerning the validity of the diffusion equation which were made earlier are still pertinent for criticality work. The diffusion equation is only an asymptotic solution of the Boltzmann equation. It will give correct results for a bare reactor if the flux is required to go to zero at a correctly chosen extrapolated boundary. Since the solutions for the various energy groups of neutrons must be required to go to zero at the same boundary, the above equations assume that the extrapolation distance is independent of neutron energy. This assumption is adequate for large reactors but may cause difficulty for the case of reactors which are not very large compared with the extrapolation distance.

Finally, it should be noted that the constant B^2 , as a property of a medium, is called the "material buckling." However, as a solution to the wave equation for a given geometry, this constant is referred to as the "geometrical buckling." In order that a reactor be just critical and that the properties of the medium be just that required by the geometry, the material buckling and the geometric buckling must be equal. In this particular case, they are both referred to as the "critical buckling." This condition forms a simple, alternative statement of the criticality equation

$$B_{\text{crit}}^2 = B_g^2 = B_m^2$$

5.2 The Adjoint Equation

Although it is a purely mathematical artifice, the concept of an adjoint flux is very useful in evaluating the effect of small disturbances on a reactor by means of perturbation theory. The adjoint equations are usually based upon the group reactor equations discussed above. If the group fluxes are represented by the vector ϕ , then the reactor equations may be written in the matrix notation

$$\mathbf{M}\phi = 0$$

The adjoint of the matrix \mathbf{M} may be formed by taking the complex conjugate of each matrix element and interchanging rows and columns. The resulting adjoint matrix \mathbf{M}^* will then have eigenfunctions analogous to the neutron flux vector ϕ . These eigenfunctions are defined by an eigenvalue equation similar to the reactor equation

$$\mathbf{M}^*\phi^* = 0$$

The eigenfunctions of the adjoint matrix are referred to as the "adjoint fluxes." The equations for the two-group adjoint fluxes are thus

$$\begin{aligned} D_1 \nabla^2 \phi_1^*(\mathbf{r}) - N(\sigma_{a1} + \xi_1 \sigma_{s1}) \phi_1^*(\mathbf{r}) + pN \xi_1 \sigma_{s1} \phi_2^*(\mathbf{r}) &= 0 \\ D_2 \nabla^2 \phi_2^*(\mathbf{r}) - N\sigma_{a2} \phi_2^*(\mathbf{r}) + \frac{k_{\infty}}{p} N\sigma_{s2} \phi_1^*(\mathbf{r}) &= 0 \end{aligned}$$

These equations may be solved in a manner entirely similar to that employed for the neutron flux equations. The resulting adjoint fluxes, however, have little direct physical significance.

For reactor work, the most important property of the adjoint fluxes and of the adjoint matrix may be shown to be (coming directly from the definition of a Hermitian conjugate or adjoint)

$$\int_V \phi_k^* \mathbf{M} \phi_k dV = \int_V \phi_k \mathbf{M}^* \phi_k^* dV$$

where ϕ_k and ϕ_k^* are two particular eigenfunctions of their respective eigenvalue equations.

This property makes possible the derivation of perturbation theory.

5.3 Perturbation Theory

Let the reactor equation matrix be perturbed by some small change ΔM in such a way that the reactor is just maintained critical. The reactor equation under this new condition is then

$$(M + \Delta M)\phi' = 0$$

where the fluxes ϕ' are those corresponding to the perturbed condition. Multiplying this equation by ϕ^* , subtracting the product of the perturbed neutron flux and the adjoint equation, and integrating the result over the volume of the reactor, the result is

$$\int_V \phi^* M \phi' dV - \int_V \phi' M^* \phi^* dV + \int_V \phi^* \Delta M \phi' dV = 0$$

The first two terms are equal in magnitude because of the properties of the adjoint fluxes and the adjoint matrix. Consequently, the perturbation equation is

$$\int_V \phi^* \Delta M \phi' dV = 0$$

This equation provides the basis for evaluating the changes which must be made, for example, in a control-rod setting to offset a change in some other reactor parameter. In essence, the perturbation relation gives the relative weighting of small effects in the reactor. If this small effect does not change the neutron flux appreciably, the unperturbed flux ϕ may be used as an approximation to the perturbed flux. The relative weighting of perturbations to a reactor may thus be stated to be the integral of the product of the adjoint flux times the neutron flux over the region of the perturbation, or

$$\int_V \phi^* \Delta M \phi dV = 0$$

For the special case of a one-group approximation, the reactor matrix M is self-adjoint, and perturbations are to be weighted as the square of the neutron flux, except for changes in the diffusion coefficient. These latter are to be weighted as the square of the flux gradient. By the same approach it can be shown that if the reactor is large (or $k - 1$ small enough), regions of different buckling in a reactor are to be weighted as the square of the flux integrated over each region of constant buckling.

6 THE FOUR-FACTOR FORMULA

The four-factor formula provides a very useful conceptual approach to neutron multiplication in a chain-reacting system. Standard formulas for calculation of the four quantities involved are well known and are presented in Art. 9 of Sec. 6-2. It is, however, necessary to realize that this physical picture is, in many cases, only a rough approximation, so that the formulas involved are sometimes inadequate for cases of engineering interest. In this section the four-factor formula will be discussed from the point of view of its validity and its application to cases where the usual approximations fail.

6.1 The Physical Concept

In the four-factor formula the physical processes within a reactor are pictured as follows:

1. All fissions in the primary fuel are caused by neutrons of thermal energy.
2. The neutrons produced in fission are all fast. The number of neutrons entering the resonance absorption region is equal to the number η of fast neutrons produced

per thermal absorption in fuel, times the fast fission factor ϵ . The fast fission factor is the number of fast neutrons slowed down below the fast fission threshold per fast neutron introduced into the system.

3. Of the neutrons entering the resonance region, a fraction p escapes resonance absorption.

4. The resulting thermal neutrons are then absorbed in fuel with a probability f , as opposed to their being absorbed in all other materials.

This neutron cycle was conceived primarily for use in graphite or heavy-water reactors and is very often adequate in reactors of these types. The neutron multiplication k_{∞} for an infinite region of the lattice in question is the product of these four factors

$$k_{\infty} = \eta \epsilon p f$$

and is a useful quantity as a first step in proceeding to calculations of finite-sized reactors.

6.2 The Fast Effect

A fraction of the neutrons produced in fission are at energies sufficiently high to produce fission in materials such as U^{238} or Th^{232} . These neutrons will also be able to produce fission in isotopes such as U^{235} , but the relative effect of these latter materials will be small, since their relative abundance in low enrichment reactors is small and since their microscopic fission cross sections at high energies are only comparable to the cross sections of other materials.

In the conventional formulas for the fast effect, it is assumed that a fission neutron will produce fissions only in the lump of fuel in which it is born. If it escapes from this lump without interaction, it is considered to be lost from the high-energy region. Corrections may be made for neutrons which leave a lump but are immediately scattered back into it. This formulation of the fast effect has been developed extensively for a number of fuel geometries.

As long as a fission neutron has a negligible chance of striking some lump of fuel besides the one in which it was born, the traditional approach to the fast effect holds satisfactorily. However, some reactors, notably those moderated with light water, have their fuel elements very close together, so that a very significant number of fissions in the high-energy region is produced by neutrons from one lump entering an adjacent lump. Where this effect is small, a correction may be derived for the interaction fast effect and may be used to modify the conventional approach. However, in many reactors of engineering interest the fuel lumps themselves are small enough and close enough together that the majority of the fast effect is produced by this interaction type of phenomenon. In this case, it has been found more successful to approach the fast effect from another standpoint.

The majority of the fast effect in a close-packed lattice is essentially that effect which would have been produced in a homogeneous mixture of fuel and moderator of the same average composition. Hence, a homogeneous fast-effect calculation can be made and can then be corrected for the slight increase in fast effect due to the heterogeneity of the lattice. This approximation has been found to give good agreement with measurements.¹⁷ Fast-effect measurements for light-water uranium lattices are much larger than would be expected from the conventional formulas.¹⁸

6.3 Resonance Escape Probability

The normal formulation of the resonance escape probability p depends almost entirely upon the evaluation of an effective absorption cross section for the fuel in its actual geometrical configuration. This effective cross section has been measured for several specific cases,¹⁹ and these measurements have given rise to the empirical surface-and-mass formula conventionally used in such calculations. This empirical approach to the calculation of resonance absorption relies upon a formula which breaks the effective resonance cross section into a strongly self-shielded surface term and a weakly self-shielded volume term. Unfortunately, the problem of arriving

at the constants in this formula, starting from fundamental cross sections, is very difficult because of the very large self-shielding of the absorption resonances.

Care must be taken in the use of the effective resonance integral formulation for resonance escape. If epithermal absorption in U^{235} is neglected, then a compensating correction must be made in the epithermal absorption properties used for U^{238} . In particular, the relative cross sections used for these two isotopes in the thermal region will adequately account for the competition between the $1/v$ portions of their absorption cross sections even in the resonance region. Thus, as a minimum, if epithermal fission in U^{235} is ignored, the $1/v$ component of the U^{238} cross section should be subtracted from the resonance integral. This treatment still ignores absorption in the U^{235} resonances which are, however, not very pronounced. Note, however, that when explicit account is taken of epithermal U^{235} absorption, the $1/v$ correction to the U^{238} mass absorption must *not* be made.

For the case of a close-packed lattice, where a neutron has a reasonable probability of traveling from one fuel lump to another fuel lump without encountering a moderator atom, the concept of surface absorption is subject to correction. In these lattices, the uranium atoms at the surface of a fuel lump are partially shielded by the uranium atoms in adjoining fuel lumps. That is, the neutron spectrum impinging on the lump is not that produced by full moderation but is partially depressed in the energy region of the strong absorption resonances by the presence of other lumps. Such an interaction correction,²⁰ similar in mathematical formulation to the correction for interaction fast effect, should be applied to all lattices which are sufficiently tightly packed.

The justification for the use of an empirical formula for resonance absorption is simply that for a heterogeneous lattice there is no other adequate way of evaluating such a quantity. In a difficult geometry the mathematical complexity is just too great. Some progress is being made, however, with the use of high-speed digital computers in the calculation of resonance escape from fundamental cross sections by use of the Monte Carlo technique.¹¹

It is to be hoped that a sound theoretical basis for investigating resonance escape in geometries of engineering interest will result from this work. It has already been reported²¹ that an energy-dependent cell calculation, using Selengut-Goertzel theory in the moderator and an appropriate boundary condition at the surface of the fuel rod, gives excellent agreement with the results of the Monte Carlo calculation. With the aid of digital computers such an approach may well provide the engineering tool that is needed.

6.4 Thermal Utilization

The fraction of thermal neutrons captured in fuel material (usually uranium) is called the thermal utilization. Calculations of the thermal utilization by conventional methods are approximate chiefly because of the use of inadequate methods for computing thermal neutron flux distributions and because of the approximate nature of the description of the energy distribution of thermal neutrons.

In many cases diffusion-theory formulas are used to calculate the disadvantage factors with which to weight cross sections. The disadvantage factor is an expression of the relative weighted average fluxes in the different materials and is dependent essentially upon the ability to calculate correctly the detailed thermal flux distribution within the reactor lattice. For many cases of interest, diffusion theory will not give this flux distribution to adequate accuracy. Various other approaches have been used to obtain adequately correct disadvantage factors. In particular, the P_3 spherical harmonics approach has been found to be extremely useful.²² The Serber-Wilson technique has also been used for lattice calculations.¹⁴ This method appears to yield improvements over diffusion theory but it is not expected to yield as accurate results as it does in criticality calculations because of the much greater importance of higher order terms in the solutions for the flux in a reactor lattice cell.

The second difficulty with calculations of the thermal utilization arises from the energy distribution of thermal neutrons. To begin with, this distribution is affected by moderator temperature and by the absorption in the lattice. If all cross sections

varied with energy in the same way (usually $1/v$), these spectrum effects on f would be relatively minor. However, the cross sections of many fuel materials are decidedly not $1/v$ and, to this extent, provide a need for the calculation of the neutron flux spectrum in some detail. This is particularly important for reactors containing appreciable amounts of plutonium because of the pronounced resonance in the cross sections of this material just above the thermal-energy region. For such materials, in order to choose cross sections correctly, it is necessary to average the energy-dependent cross-section curves over the neutron distribution.

The calculation of thermal-neutron spectra is exceedingly complex because of the effect of molecular binding on the scattering atoms. To date most calculations have dealt with gaseous moderators and no adequate treatment is available for the cases of real interest. Methods for getting a temperature characteristic of some equivalent Maxwellian distribution are often used. This effective temperature problem is discussed in Art. 7, Special Reactor Problems.

A further complication in the determination of neutron spectra for use in thermal utilization calculations is the fact that the spectrum is not constant throughout the lattice but will be of somewhat higher energy (usually referred to as "hardened") in the regions of low moderation and high absorption. Both for the choice of cross sections and for the calculation of the flux distribution, it is the neutron spectrum locally incident upon the atoms in question that is desired. Hence, it is necessary, ideally, to know the spectrum at each point in the lattice. Calculations of these hardening effects have been made, but generally the variations of the neutron spectrum throughout a lattice are not of first-order importance and are usually neglected in reactor calculations, except for reactors of unusual type.

6.5 Fast Neutrons per Thermal Absorption

The calculation of the number of fast neutrons produced per thermal neutron absorbed in fuel is usually straightforward. It is made complex only by the fact that the ratio of capture to fission in fissionable materials varies slightly with energy. Thus, η is somewhat dependent upon the neutron spectrum. For materials other than plutonium this variation is relatively minor. For plutonium the presence of the 0.3 eV resonance, with its accompanying change in the value of capture-to-fission, again demands a considerable knowledge of the thermal neutron spectrum in order to obtain a satisfactory value for η . The value of η for Pu^{239} , for example, is given as 2.03 at 2,200-m/sec neutron velocity.²³ However, because η is lower in the large resonance at 0.3 eV, the value of η for Pu^{239} in a typical power reactor will be about 1.85.

6.6 Inadequacies in the Formulation

The above discussion has primarily involved improved methods for calculating the various components of the four-factor formula. It is important to realize, however, that the four-factor formula itself contains some important inadequacies arising directly from the physical picture involved. To begin with, the fast effect will probably be dependent upon reactor size for small, close-packed reactors. The fast neutron leakage correction which is made to k_{∞} thus probably does not adequately give the effect on k_{∞} caused by the loss of very fast neutrons. This effect, however, should be relatively minor.

The major inadequacy in the four-factor formula is the omission of nonthermal fissions. In many reactors an appreciable fraction of the fissions is produced by neutrons of epithermal energies. This will be particularly true in close-packed light-water lattices and is even true to some extent in the more diffuse graphite or heavy-water lattices. It should be noted, however, that the addition of resonance or epithermal fission terms to the conventional reactor formulations must be accompanied by associated corrections to resonance absorption in U^{238} (Art. 6.3). For reactors containing a small percentage of epithermal fission, the inclusion of resonance fission will cause relatively minor changes in calculated reactor characteristics. However,

if epithermal fissions are a significant fraction of the total number of fissions, then the conversion ratio and the reactivity may be appreciably affected by the different competition between materials in the epithermal and resonance regions and the fact that the epithermal value of η for U^{235} (and for the plutonium isotopes as well) is much less favorable.

7 SPECIAL REACTOR PROBLEMS

In the application of many of the methods discussed in this handbook there are special effects which may have important consequences in actual reactor operation. A few of these effects will be discussed in this section in an effort to provide a better understanding of the behavior of the nuclear systems involved.

7.1 Thermal Spectrum Effects

In many reactor calculations the low-energy neutrons are treated as a single energy group of thermal neutrons. Cross sections for this group must be obtained by averaging cross sections over the thermal-energy spectrum in order to arrive at physically significant results. However, in such a treatment of the thermal group it is usually necessary to take the characteristic temperature of the neutron distribution as that of the moderator molecules with which the neutrons are in energy equilibrium. For fairly dilute systems which are geometrically homogeneous this approximation is entirely adequate. There are important cases where this treatment fails, however. In heterogeneous geometries, where the lumps of fuel are of appreciable size compared with the mean free path of a neutron within the lump, the effective neutron spectrum impinging on the fuel atoms varies with position. The neutrons at lower energies are more strongly absorbed, with the result that the remaining spectrum is composed of neutrons with a higher effective "temperature." This phenomenon is usually described as "absorption hardening" and may be of importance in describing the characteristics of heterogeneous reactors with large fuel elements. However, the present trend toward small fuel elements (for engineering reasons) will avoid some of the corrections due to this effect. Of course, clumped, small fuel elements (for example, in a water-graphite or sodium-graphite reactor) will again exhibit appreciable neutron spectral hardening in the absorption in the fuel.

Perhaps a more important aspect of the absorption hardening problem arises in both heterogeneous and homogeneous reactor lattices which contain strong thermal absorption. This applies particularly to light-water lattices, where the relative amount of thermal absorption is high because of the large amount of fuel present. In these cases the absorption hardening effect is a characteristic of the entire lattice rather than of a restricted geometrical region. The increase in the effective temperature of neutrons for a homogeneous reactor moderated with gaseous hydrogen has been studied by Wigner and Wilkins²⁴ and for a reactor with a moderator of heavy elements by Wilkins.²⁵ A useful approximation to the results of these calculations has been suggested by Cohen²⁶, who found that the results were roughly of the form

$$\frac{\bar{v}}{v_{kT}} = 1.128 + 1.36 \frac{\Sigma_a^{kT}}{\xi \Sigma_s}$$

where \bar{v} = average velocity of hardened "Maxwellian"

v_{kT} = velocity corresponding to thermal temperature

Σ_a^{kT} = absorption cross section at v_{kT} (assumed to be $\sim 1/v$)

$\xi \Sigma_s$ = scattering properties of the medium (assumed independent of velocity)

The resulting average velocity may then be used to obtain an effective kT for use in the usual cross section averaging techniques. This empirical formula has been checked roughly by the work of various experimenters. More recent work has been concentrated on treating the problem of nongaseous moderators.²⁷ To date results of engineering significance have not been achieved for liquid moderators.

7.2 Self-absorbed Cross Sections

The depression in neutron flux in the region of a high-absorption cross section has been discussed elsewhere. One aspect of this problem, however, deserves special mention. In the calculation of thermal utilization in a heterogeneous reactor the various absorption cross sections are weighted by their geometrical average fluxes to take into account the depression of the flux, for example, in a fuel rod. The effective cross section used may be written (where ϕ_i is the average flux in material i)

$$\Sigma_{eff} = \frac{\sum_i (N\sigma)_i \phi_i}{\sum_i \phi_i}$$

In many calculations, however, it is inconvenient to perform such a space-wise flux weighting as a part of reactivity calculations. This is particularly true in multigroup work. Consequently, it is often desirable to regroup the above expression:

$$\Sigma_{eff} = \frac{\sum_i N_i (\sigma \phi)_i}{\sum_i \phi_i} = \sum_i N_i \sigma_{eff,i}$$

The quantity Σ_{eff} may be thought of as the effective absorption cross section in a homogeneous material having the same over-all nuclear properties as the actual heterogeneous region under consideration. The cross section $\sigma_{eff,i}$ is often referred to as self-shielded or self-absorbed.

An excellent example of this kind of thinking comes in the conventional treatment of resonance escape. Uranium, which has a resonance integral of 240 barns in a dilute homogeneous system, is self-shielded in the energy range of the absorption resonances because of the strong absorption at the surface of the lump of neutrons at resonance energy. The resulting effective resonance cross section, given by the empirical formula, is usually only about 10 barns.

The Dancoff and Ginsberg self-shielding correction to the resonance escape integral,²⁰ which is mentioned in the discussion of resonance escape in this section, is another form of self-absorption correction. It is effectively a correction to the surface area of fuel element which is available to neutrons, but it arises primarily as a depression of the incident flux.

7.3 The Doppler Effect

Most neutron interaction calculations picture the collision process as being made up of a neutron of some given velocity impinging upon a scattering or absorption atom which is at rest. In actuality this picture is not correct, since the target atom is in thermal motion in the atomic or molecular system of which it is a part. Because there are components of this thermal motion in the direction of the incident neutron, the effective energy of the neutron becomes a spread-out distribution about the actual neutron velocity with respect to the average position of the target atom. This phenomenon is entirely analogous to the Doppler broadening of lines in optical spectroscopy or to the shift of acoustic frequency due to a motion of the source of sound with respect to the observer. For this reason it is referred to as a Doppler effect.

It can be shown, for a Maxwellian distribution of target atom velocities and a $1/v$ cross section, that the motion of the target nucleus makes no difference in the effective cross section of the interaction. Although this condition covers a very large fraction of the cases of interest, certain cross sections which depart widely from $1/v$ will have a noticeable change due to the Doppler effect. In particular, for resonances the effect can be important. It may be shown that the area under a Breit-Wigner resonance

is not changed by this target atom motion, so that in very dilute systems there is again no reactivity change due to Doppler effect. However, in many cases of interest the resonance peaks may be strongly self-shielded. The vibration of the target atom (which lowers the effective resonance peak height but spreads it out, thus maintaining the same area under the peak) cannot change the effective cross section inside the resonance itself, since the absorption is already saturated. However, the widening of the resonance will cause a widening of the region of strong capture, so that absorption in the skirts of the resonance will increase. This effect is referred to as "Doppler broadening" of the resonance. It is of importance to reactor operation.

In low-enrichment reactors and, indeed, in many others, resonance absorption is less desirable than thermal absorption because of a lower probability of fission. Hence Doppler broadening will reduce reactivity with increased temperature. That is, Doppler broadening produces a negative temperature coefficient. In most reactors this negative coefficient is in the range of 1 to $2 \times 10^{-5}(\Delta k/k)/^{\circ}\text{C}$. This is a fairly small temperature coefficient in most systems. It is still important, however, since it is prompt. In a reactor power excursion the removal of reactivity due to this temperature coefficient does not lag, since there is no necessity for a heat-transfer process to take place. Since fission energy is largely dissipated directly in the fuel material by the passage of the fission fragments, the temperature of the fuel rises immediately and the Doppler broadening takes effect promptly. Because of this, although other temperature coefficients may be larger, the Doppler broadening is important to reactor dynamics and safety.

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6-2 REACTOR CALCULATIONS

BY

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Nomenclature

(See also Table 6 for special symbols used in Arts. 7 and 8)

Complete definitions of the symbols may be found in the sections referred to.

- A = mass number
 B^2 = buckling. The geometrical buckling of a medium is that which satisfies the equation $\nabla^2\phi + B^2\phi = 0$ in the medium with proper boundary conditions. The material buckling is that which satisfies the equation $-L^2B^2 - 1 + (k/p)F_\infty(E_s, B^2) = 0$ (Art. 5.3)
 B'^2 = in the two-group representation, the "transient" buckling (see Table 6)
 D = diffusion coefficient (Art. 2.13)
 E = neutron energy
 $F(E)$ = collision density per unit energy (Art. 3.2)
 f = thermal utilization (Art. 9.2)
 $G(x, x')$ = diffusion kernel for thermal neutrons = flux at x due to unit source at x' (Art. 2.25)
 $I_0, I_1, J_0, J_1, K_0, K_1, Y_0, Y_1$ } = Bessel functions
 J = neutron current density (Art. 2.11)
 k = Boltzmann constant
 k = multiplication constant (infinite) (Art. 9.1)
 k_{eff} = effective multiplication constant in a finite reactor (Art. 13.1)
 k_{ex} = excess multiplication factor (Art. 13.1)
 L = diffusion length = $\sqrt{D/\Sigma_a}$ (Art. 2.15)
 L_f = slowing-down length (Art. 5.6)
 l = prompt neutron lifetime (Art. 13.3)
 \mathcal{L} = neutron leakage = fraction of neutrons produced which leak from reactor
 M = migration length = $\sqrt{L^2 + \tau}$ (Art. 5.7)
 n = neutron density, neutrons/cm³
 N = number of nuclei per cm³
 N = Avogadro number = number of atoms per gram atomic weight
 p = resonance escape probability (Art. 9.4)
 $P(E, r, r')$ = finite slowing-down kernel (Art. 5.1)
 $P_\infty(E, |r - r'|)$ = infinite slowing-down kernel (Art. 5.2)
 $\bar{P}_\infty(E, B^2)$ = three-dimensional Fourier transform of the infinite slowing-down kernel (Art. 5.3)
 $q(E)$ = slowing-down density (Art. 3.2)
 Q, q = neutron source strength, neutrons/sec or neutrons/(cm³)(sec)
 R = radius of a spherical or cylindrical medium

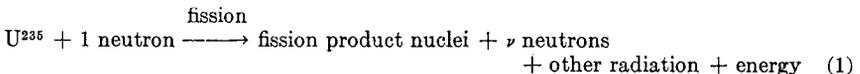
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- S = in the two-group representation, the ratio ϕ_f/ϕ_s (see Table 6)
 t = time variable
 T = thickness of slab medium
 T = absolute temperature
 v = neutron velocity
 V = volume
 α = most probable neutron velocity in a Maxwell-Boltzmann distribution (Art. 4.1)
 β = albedo = ratio of current density out of medium to current density into medium [if net current is into medium (Art. 2.3)]
 β = fraction of fission neutrons originating from delayed emitters (Art. 13.2)
 β_i = fraction of fission neutrons originating from delayed emitters of i th species (Art. 13.2)
 γ = extrapolation distance measured in transport mean free paths (Art. 2.22)
 ϵ = fast fission factor (Art. 9.5)
 η = regeneration factor = average number of fission neutrons produced per neutron absorbed by fissionable material (Art. 9.2)
 κ = reciprocal of diffusion length ($1/L$)
 κ_s = reciprocal of diffusion length
 κ_f = reciprocal of slowing-down length ($1/L_f$)
 λ = mean free path of neutron = $1/\Sigma$
 λ_i = decay constant of i th species of delayed-neutron emitter (Art. 13.2)
 $\bar{\mu}_0$ = average cosine of scattering angle per collision of neutron with scattering nucleus (laboratory coordinate system) (Art. 2.13)
 ν = average number of neutrons emitted per fission
 ξ = average logarithmic change in energy per collision (Art. 3.1)
 ρ = reactivity = $k_{\text{eff}}/k_{\text{eff}}$ (Art. 13.1)
 ρ = density, g/cm³
 σ = microscopic cross section, cm²/nucleus
 σ_a = microscopic absorption cross section
 σ_s = microscopic scattering cross section
 σ_{kT} = cross section corresponding to most probable velocity in Maxwell-Boltzmann distribution at temperature T (Art. 4.2)
 Σ = macroscopic cross section, cm⁻¹
 Σ_a = macroscopic absorption cross section
 Σ_s = macroscopic scattering cross section
 $\Sigma_s; \Sigma_{as}$ = macroscopic absorption cross section for thermal group of neutrons in two-group representation (see Table 6)
 $\Sigma_f; \Sigma_{af}$ = macroscopic slowing-down cross section for fast group of neutrons in two-group representation (see Table 6, also Art. 5.6)
 $\tau(E)$ = Fermi age from source energy to energy E (Art. 3.4)
 ϕ = neutron flux density (Art. 1.3)
 $\phi_{th}; \phi_s$ = flux density of thermal neutrons
 ϕ_f = flux density of fast or epithermal neutrons
 ϕ^* = adjoint function (Art. 14.1)

1 INTRODUCTION

1.1 The Fission Reaction

The basic process which makes possible the nuclear reactor is the fission of heavy nuclei by neutrons. The products of a given single fission cannot be predicted, but if many fissions are considered, an average equation for the reaction can be written for a given fissionable isotope; for example,



The significant point for the present discussion is that the average number ν of new neutrons produced per fission is considerably greater than unity (actually about 2.5). The fission chain reaction in a nuclear reactor is maintained at a steady rate if the materials of the reactor are so arranged that exactly one of the ν neutrons produced by each typical fission interacts with another fissionable nucleus to produce still another fission. If, on the average, less than one of the fission neutrons produces a further fission, the neutron population of the reactor will decrease continuously and the chain reaction will eventually die out. If, on the other hand, more than one of the neutrons from the average fission produces further fissions, the neutron population of the reactor will grow from generation to generation and the level of fission power production in the reactor will increase continuously with time. Most problems in reactor physics involve either the determination of the necessary conditions for a steady-state chain reaction (criticality conditions) or the determination of the rate of growth or decay of the chain reaction (reactor kinetics) under given reactor conditions. In either case, the statics or kinetics of the chain reaction is determined by the statics or kinetics of the neutron population of the reactor. The problem is usually solved by investigating the life histories and reproductive powers of "average" neutrons. Evidently, then, the subject of reactor physics consists mainly of neutron physics or, more specifically, of the physics of neutron transport and neutron interactions with matter.

1.2 Fundamental Neutron Processes

The individual neutron can exist for only a short time as a free particle. It ceases to exist as a particle when it enters the nucleus of an atom and becomes integrated into the structure of that nucleus. When such an event occurs, the neutron is said to have been absorbed by the nucleus. The resulting modified nucleus may be stable or may be unstable and emit other particles, including neutrons (as in the case of fission), but in any case the original neutron is considered to have ceased to exist. Aside from absorption, which ends its life, the neutron can interact with matter only by making scattering collisions with the nuclei which compose the matter. These interactions are called *collisions* because the free neutrons in a material medium are normally traveling about, in the spaces between nuclei, at high velocity. The velocities may have been imparted to the neutrons when they were formed or may be the velocities of thermal agitation. In a scattering collision the velocity of the colliding neutron is usually changed, both in magnitude and in direction. If the neutron imparts only kinetic energy to the body with which it collides, the collision is said to be elastic. If it changes also the internal energy of the target body, the collision is said to be inelastic.

1.3 Quantitative Specification of Interaction Rate: Cross Sections

Experimentally it is found that the number of interactions of any given type (scattering, absorption, fission, etc.) which occur per second per unit volume of a material containing neutrons depends, for a material of given nuclear species, only on the number of nuclei per unit volume, the number of neutrons per unit volume, and the effective neutron velocity. This relationship is formulated quantitatively by supposing that each nucleus of a given species presents a target for a given type of interaction of area $\sigma(v)$ cm² to any neutron traveling with velocity v (the neutron itself being considered a dimensionless point). On this basis a given neutron, in the time interval Δt , will travel a distance $v \Delta t$ cm and will experience interactions of the specified type with all nuclei which lie within the volume $\sigma(v)v \Delta t$ cm³. If there is a uniform density of N nuclei per cubic centimeter, the number within this volume is just $N\sigma(v)v \Delta t$. If the neutron density within the medium is n neutrons per cubic centimeter, then the rate at which interactions of the specified type are occurring per unit volume is

$$\text{No. of interactions}/(\text{cm}^3)(\text{sec}) = nvN\sigma(v) \quad (2)$$

The quantity nv is called the neutron flux. It is often represented by the single

symbol ϕ . Its dimensions are neutrons per square centimeter per second. The quantity σ is called the microscopic cross section for the type of interaction under consideration. Its dimensions are square centimeters per nucleus. Microscopic cross sections are usually quoted in *barns*. The barn is a unit of area equal to 10^{-24} cm². However, in the equations presented here the units of σ are square centimeters unless otherwise specified. The quantity $N\sigma$ is called the macroscopic cross section. It is the total target area for a given interaction which is presented to a neutron by all the nuclei in a cubic centimeter of material; its dimensions are cm² per cm³, or cm⁻¹. It is often represented by the single symbol Σ .

The average number of interactions made by a neutron in traveling 1 cm is just $N\sigma = \Sigma$. Conversely, the average distance traveled by a neutron between interactions is $1/\Sigma$. This distance is called the mean free path for the type of interaction under consideration and is usually designated by the symbol λ .

As implied above, a microscopic cross section is assigned to each nuclear species for each type of neutron interaction, and in general the cross section is a function of neutron velocity (or, more precisely, of the relative velocity of the neutron with respect to the interacting nucleus). Although from the standpoint of nuclear physics the cross sections for the various types of interactions are not necessarily independent of one another, for the purposes of neutron physics they may be and are considered to be so.

Evidently the concept of the cross section is not a particularly useful one unless the cross section is a property of the nucleus alone, unaffected by such considerations as the molecular or crystal structure of the material in question. This condition is met in most cases which arise in elementary reactor theory. When it is met, any material may be regarded simply as a mixture of the various nuclear species present, and the macroscopic cross section for any given type of neutron interaction is just the sum of the macroscopic cross sections of all the nuclear species for that type of interaction; i.e.,

$$\Sigma = \sum_i N_i \sigma_i = \sum_i \frac{\rho_i}{A_i} N \sigma_i \quad (3)$$

where N_i is the number of nuclei of the i th species per cubic centimeter of the medium, ρ_i is the number of grams of the i th species per cubic centimeter of medium, A_i is the atomic weight of the i th species, and N is the Avogadro number.

1.4 Sizes of Cross Sections and Other Important Dimensions

Scattering cross sections for neutrons of energies important in reactor physics range from about 2×10^{-24} to about 20×10^{-24} cm². Absorption cross sections cover a much wider range of values. For neutrons of thermal velocities the absorption cross section of hydrogen is about 0.3×10^{-24} cm², that of iron is about 2×10^{-24} cm², and that of the boron isotope of mass 10 is about 4×10^{-21} cm². If the cross section is visualized as circular in shape (i.e., as the cross section of a sphere), the radius corresponding to a cross section of, say, 10×10^{-24} cm² is about 2×10^{-12} cm. The average distance between nuclei in typical solids or liquids is very much larger, of the order 10^{-8} cm. That is to say, even though many nuclei are packed into each cubic centimeter of material, the targets which they present to a neutron for interaction are so small that the neutron may, in typical cases, travel distances of the order of centimeters between interactions. Consequently, one of the problems in maintaining a neutron chain reaction is simply to contain a large fraction of the neutrons in the reactor long enough for them to interact with the fissionable material. The wandering away of neutrons before they can be absorbed is generally referred to as leakage from the reactor.

Despite the relatively long distances traveled between interactions, typical neutrons have such high velocities that they exist for only a short time before being absorbed. The mean lifetime of a neutron in a reactor depends upon the reactor design but is seldom more than 10^{-3} sec. It is not possible to maintain a high density of neutrons

in the face of such a short lifetime unless the rate of production of neutrons per unit volume by fission is very high, i.e., unless the power production per unit volume is very high. In the reactors built to date it has not been possible to achieve power densities which give neutron densities greater than about 10^9 neutrons/cm³. Since the atomic density in typical solids or liquids is of the order 10^{23} atoms/cm³, it is evident that the probability that a neutron will encounter an atomic nucleus is enormously greater than the probability that it will encounter another neutron. Because of this circumstance, that every free neutron in the medium acts independently of all other free neutrons, the equations which describe the flow of neutrons in a medium are linear. Some of the consequences of this linearity are discussed in Art. 2.25.

1.5 Conditions for the Chain Reaction

As stated in Art. 1.1, a steady self-sustaining chain reaction is maintained in a nuclear reactor if the materials of the reactor are so arranged that exactly one of the ν neutrons produced by each average fission reacts with an atom of fissionable material to produce another fission. The subsequent articles have indicated that two processes compete with the fissionable nuclei for the free neutrons: absorption by nonfissionable nuclei and leakage from the reactor. The condition for criticality of the reactor can be written in terms of these two processes if two new symbols are defined:
 \mathcal{L} = fraction of neutrons produced which leak from the reactor

$$k = \frac{\text{No. of second-generation neutrons produced}}{\text{No. of first-generation neutrons absorbed (by all materials present, including fissionable material)}}$$

The equation for criticality is, then,

$$k(1 - \mathcal{L}) = 1 \quad (4)$$

The quantity k can be stated in a more significant way if another symbol η is introduced. The quantity η is the average number of fission neutrons produced per neutron absorbed in fissionable material. In general η is less than ν . The expression for k becomes

$$k = \eta \frac{\text{No. of neutrons absorbed in fissionable material}}{\text{total No. of neutrons absorbed}}$$

According to Eq. (4), k must be made relatively large and \mathcal{L} must be made relatively small if the chain reaction is to be self-sustaining. The maximum value of k is attained if the reactor is made of pure, fissionable isotope. Such a reactor is of little use as a power source, since no effective means is supplied for removing the heat which is generated and converting it to useful work. Practical power reactors must contain materials other than the fissionable isotope. These materials serve the purposes, for example, of coolants, structural materials, and cladding to prevent the escape of fission products from the fissionable material and, in some cases, to protect the fissionable material from chemical attack by other materials. Furthermore, in many cases fertile materials— isotopes which can be converted to fissionable isotopes by the absorption of neutrons—are included in the reactor. Consequently, in many cases the practically attainable value of k is not much greater than unity, and \mathcal{L} must be made quite small in order to achieve criticality. This requirement is particularly demanding if the reactor is to be fueled with the naturally occurring mixture of the fissionable isotope U²³⁵ with the fertile isotope U²³⁸.

One way of decreasing \mathcal{L} —of decreasing the probability that a neutron will wander out of the reactor—is to make the reactor large. Another method, which is more economical, is to include in the reactor a good moderating material: a material of low mass number, low absorption cross section, and relatively high scattering cross section. The neutrons, in elastic collisions with the moderator nuclei, are slowed down to relatively low velocities. Since absorption and fission cross sections are much larger for slow than for fast neutrons, the moderation, or slowing down, of the

neutrons increases their probability of absorption and hence decreases the leakage probability. Most frequently, in a moderated reactor, the amount of moderator employed is so large that most neutrons come into energy equilibrium with the thermal agitation of the moderator nuclei before they are absorbed. Such a reactor is called a thermal reactor.

Reduction of leakage is not, of course, the only way in which neutron moderation affects the criticality condition. Since fission and capture cross sections vary with neutron energy in different ways, the value of k which applies for a given mixture of absorbing and fissionable isotopes depends upon the energy spectrum of the neutrons present. This dependence is strikingly illustrated by the fact that whereas it is quite feasible to construct a critical thermal reactor which employs as fuel the naturally occurring mixture of U^{235} and U^{238} , the value of k for a large block of pure natural uranium (no moderator added) is considerably less than unity.

The methods outlined in this subsection apply primarily to thermal reactors, although much of the basic material, as well as the specific methods of Art. 10, are more generally applicable.

1.6 Outline of Subsequent Treatment of Reactor Physics

To summarize the foregoing, the behavior of a nuclear reactor is determined by the migrations of neutrons, their slowing down, and the relative rates at which they are absorbed by fissionable and nonfissionable isotopes. When these processes are such that the neutron population of the reactor remains constant in time, the reactor is said to be critical; the study of the conditions for criticality constitutes the subject of reactor statics. The behavior of the neutron population when the criticality condition is not fulfilled constitutes the subject of reactor kinetics. The remainder of the subsection is devoted principally to reactor statics; further material on kinetics is contained in Sec. 8-1.

The material in the remainder of this subsection may be summarized as follows: Article 2 treats the flow of neutrons of uniform energy in a medium. Article 3 treats the slowing down of neutrons to thermal energy, and Art. 4 describes their distribution in energy after they have reached thermal equilibrium in the moderator. In Art. 5 the criticality condition for a reactor is stated quantitatively as a partial differential equation in terms of the fundamental processes covered in the preceding sections. Article 5 amounts to a quantitative statement for the leakage \mathcal{L} of Eq. (4). Article 6 gives solutions for the critical reactor equation in simple geometries, and Arts. 7 and 8 outline methods of solution for the more complex cases of reflected reactors. Article 9 covers the evaluation of the constants which enter the reactor equation, including the multiplication constant k [Eq. (4)]. This article covers both homogeneous cases and the more complex "lumped" cases. Article 10 outlines briefly the multigroup technique which can be used for both thermal and nonthermal reactors. In Art. 11 typical computation procedures are illustrated by specific examples. Article 12 treats the effects of lumped thermal-neutron absorbers, such as control rods or experimental samples, in a reactor. Article 13, on noncritical reactors, is a very brief treatment of reactor kinetics and concepts related to reactor kinetics. In Art. 14 relations are given by which the reactivity effects of small changes in a previously critical reactor may be evaluated.

Throughout the subsection an attempt has been made to limit the coverage to those items appropriate to handbook treatment. Individual computations which would normally take more than an hour or two of work on a desk-type calculating machine or a slide rule for their execution are considered to be beyond the scope of a handbook. The treatment of neutron migration has been limited to the diffusion approximation. In general, where specific approximations are used, they are noted. A complete discussion of the implications of such approximations and of their range of applicability is evidently beyond the scope of the subsection. Finally, although an effort has been made to arrange the topics in a logical sequence and to impart as much physical understanding as can be done in a brief space, the presentation is obviously not intended to approximate a logical and integrated development of reactor theory.

It is intended, rather, that the subsection constitute a convenient reference for those elementary formulas and procedures which occur frequently in the course of reactor computation.

2 DIFFUSION OF "MONOENERGETIC" NEUTRONS

Conceptually, the material of this article applies to the diffusion of monoenergetic neutrons; practically, it is applied to groups of neutrons whose energy spread is narrow enough and predictable enough that averaged values of properties of the medium may be used. The most common case is that of thermal neutrons. The averaging of diffusion properties over the thermal-energy spectrum is treated in Art. 4.

2.1 The Diffusion Equation

2.11 Current Density. The current density J is the *net* number of neutrons flowing in unit time through a unit area normal to the direction of flow.

$$J = -D \text{grad } \phi \quad (5)$$

or, in Cartesian coordinates,

$$J = -D \left(i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right) \quad (6)$$

where D is the diffusion coefficient and i , j , and k are, respectively, the cosines of the angles between the x , y , and z directions and the normal to the plane across which current density is being measured. That is to say, the net current density across the plane has directional components J_x , J_y , and J_z , given by

$$J_x = -D \frac{\partial \phi}{\partial x} \quad J_y = -D \frac{\partial \phi}{\partial y} \quad J_z = -D \frac{\partial \phi}{\partial z}$$

Each component (e.g., J_x) of the *net* current density is the difference between a forward current density component J_{x+} and a reverse current density component J_{x-} , given by

$$J_{x+} = \frac{\phi}{4} - \frac{D}{2} \frac{\partial \phi}{\partial x} \quad J_{x-} = \frac{\phi}{4} + \frac{D}{2} \frac{\partial \phi}{\partial x} \quad (7)$$

with similar expressions for the other directions.

2.12 Neutron Leakage Rate. The rate at which neutrons leak from unit volume in a diffusion medium is

$$\begin{aligned} \text{Neutron leakage}/(\text{unit vol})(\text{sec}) &= -D \nabla^2 \phi \\ &= -D \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) \end{aligned} \quad (8)$$

in cartesian coordinates where ∇^2 is the Laplacian operator (see Sec. 3-2 for other coordinate systems).

2.13 Evaluation of Diffusion Coefficient D . If experimentally determined values of D are not available, the following relation (from transport theory) may be used:

$$D = \frac{1}{3\Sigma(1 - \bar{\mu}_0) \left(1 - \frac{4}{5} \frac{\Sigma_a}{\Sigma} + \frac{\Sigma_a}{\Sigma} \frac{\bar{\mu}_0}{1 - \bar{\mu}_0} + \dots \right)} \quad (9)$$

where Σ and Σ_a are the total and absorption macroscopic cross sections, respectively, and $\bar{\mu}_0$ is the average cosine of the scattering angle per collision (stationary coordinate system).

If the medium absorbs only very weakly, the expression becomes

$$D = \frac{1}{3\Sigma_s(1 - \bar{\mu}_0)} = \frac{\lambda_{tr}}{3} \quad (10)$$

where λ_{tr} is the transport mean free path.

In reactor systems, scattering of neutrons is essentially isotropic in the center of mass coordinate system, and for this condition

$$\bar{\mu}_0 = \overline{\cos \psi} = \frac{2}{3A} \quad (11)$$

where ψ is the scattering angle in the stationary (laboratory) coordinate system and A is the mass number of the diffusion medium.

2.14 Neutron Balance per Unit Volume—The Diffusion Equation. The rate of change of neutron density n (= number of neutrons per cubic centimeter) is given by

$$\begin{aligned} \frac{dn}{dt} &= \text{rate of leakage into unit vol} - \text{rate of absorption/unit vol} \\ &\quad + \text{rate of production/unit vol} \\ &= D \nabla^2 \phi - \Sigma_a \phi + S(r) \end{aligned} \quad (12)$$

where $S(r)$ is the effective source strength, or effective rate of production of neutrons per unit volume. The quantity $S(r)$ is smaller than the actual source strength by the factor $[1 - \frac{1}{6}(\Sigma_a/\Sigma)]$ because of the breakdown of diffusion theory near the source.

2.15 Source-free, Steady-state Diffusion Equation; Diffusion Length. In any portion of a medium which contains no sources, the steady-state diffusion equation becomes

$$\begin{aligned} D \nabla^2 \phi - \Sigma_a \phi &= 0 \\ \nabla^2 \phi - \kappa^2 \phi &= 0 \end{aligned} \quad (13)$$

generally written

where $\kappa^2 = \Sigma_a/D$. The reciprocal of κ^2 , D/Σ_a , is called the diffusion area and is represented by the symbol L^2 , which can be shown to be one-sixth the mean square distance (crow flight) traveled by a neutron in a diffusion medium between the time of its birth and the time of its absorption. The square root L of the diffusion area is called the diffusion length.

The diffusion equation, when written in the form of Eq. (13), is frequently referred to as the wave equation.

2.2 Solution of the Diffusion Equation

2.21 General Considerations. The solution of the steady-state diffusion equation in an infinite medium can be written out formally, for the general case of an arbitrary distribution of sources, by use of the diffusion kernels (see below). The solution for a finite medium is difficult unless the source distribution and the boundaries display a reasonable degree of symmetry. A few useful cases of relative simplicity are treated in following paragraphs. Other cases are covered later (Art. 12) in connection with lumped absorbers in the reactor. Solutions for a great many cases are given in a paper by Wallace.^{1,†}

Before the solutions can be given, it is necessary to consider the boundary conditions which apply at the edges of finite media. The diffusion equation does not describe accurately the flux behavior near boundaries. If, however, appropriate boundary conditions, derived by transport theory, are applied, the solutions of the diffusion equation will describe the distribution of flux adequately in all parts of a finite medium except those within about one mean free path of the boundary. The conditions for a number of cases are given below.

2.22 "Black" Boundaries. A boundary is called black if no neutrons which cross the boundary from the medium return to the medium. External boundaries between the medium and vacuum are black. Boundaries between the medium and very strong absorbers, either internal or external, may often be considered black for practical purposes, the criteria being that the absorber be thick enough to reduce the neutron flux practically to zero in its interior and that $\Sigma_a \gg \Sigma_s$ in the absorber. Evidently, the neutron flux in the medium will decrease as the boundary is approached. The boundary condition depends on the geometry of the boundary and on the transport

† Superscript numbers refer to References at end of subsection.

mean free path in the medium. It is generally stated either as the value of the logarithmic derivative of the flux at the boundary

$$\left[\frac{1}{\phi} \frac{d\phi}{dx} \right]_{\text{boundary}} = - \frac{1}{\gamma \lambda_{tr}} \quad (14)$$

where x is the coordinate normal to the boundary (the positive direction being from medium to boundary) and γ is a number characteristic of the shape of the boundary, or as the extrapolation (or augmentation) distance ϵ :

$$\epsilon = \gamma \lambda_{tr} \quad (15)$$

The extrapolation distance is defined as the distance beyond the boundary at which the flux would become zero if its normal derivative at the boundary were extrapolated linearly. The two methods of statement of the boundary condition are equivalent. A convenient way of applying the condition of Eq. (15) is simply to set up a fictitious boundary (the extrapolated boundary) at a distance ϵ beyond the true boundary and require that the neutron flux vanish at this extrapolated boundary. This boundary

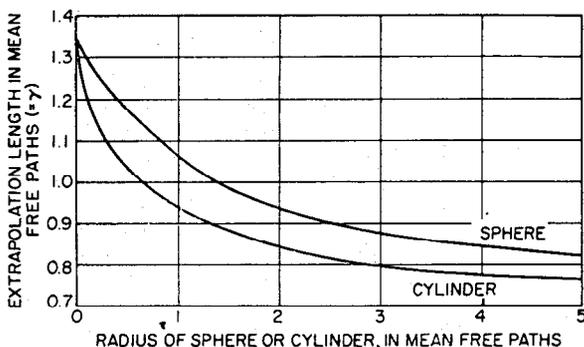


FIG. 1. Estimated values of linear extrapolation length for "black" spheres and cylinders. (Reproduced from: B. Davison and S. Kushneriuk, *Linear Extrapolation Length for a Black Cylinder*, National Research Council of Canada, Division of Atomic Energy, MT-214, March 30, 1946.)

condition is not strictly equivalent to that of Eq. (14) (since the solution of the diffusion equation is not, in general, linear between the real and extrapolated boundaries), but it is adequate for almost all practical cases. Values of γ for various geometries are given below. These apply for media of low absorption (strictly, for non-absorbing media).

Plane Boundary. $\gamma = 0.71$.

Internal Cylindrical and Spherical Boundaries. The extrapolation distance varies with the curvature; γ ranges from $\frac{2}{3}$ for vanishingly small radius to 0.71 for infinite radius. Figure 1 shows the probable dependence of γ on radius of black absorber as estimated by Davison and Kushneriuk.

2.23 "Gray" Boundaries. A strong absorber which yet does not absorb all the neutrons incident upon it is often referred to as a "gray" absorber. Diffusion theory does not describe the neutron flux behavior accurately in the vicinity of such an absorber. Kushneriuk and McKay have derived extrapolation distances, by approximations to transport theory, for "gray" cylinders embedded in nonabsorbing, scattering media (Fig. 2). These extrapolation distances may be used with diffusion theory, in the same way as the black boundary extrapolation distances, to determine the flux distributions in the scattering medium.

2.24 Boundaries between Nonabsorbing (or Weakly Absorbing) Media. The boundary conditions between two diffusion media are

1. The neutron flux is continuous across the boundary.

2. The neutron current density in a direction normal to the boundary is continuous across the boundary.

If the two media are nonabsorbing, the presence of the interface does not affect the adequacy of the diffusion approximation. If either or both of the media absorb, higher order approximations to transport theory are needed to describe the neutron distribution adequately.² Nevertheless, the diffusion approximation, with the above

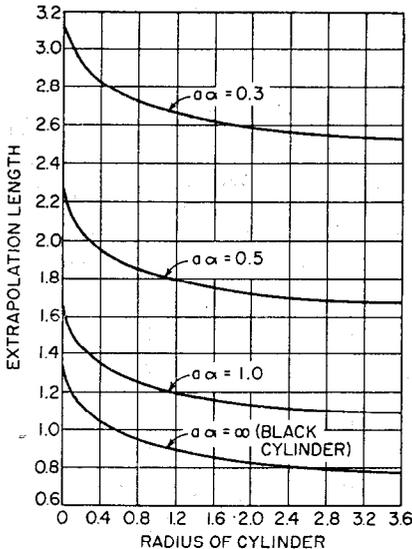


FIG. 2. Linear extrapolation length for a "gray" purely absorbing cylinder in a non-capturing medium. The unit of length for both radius of cylinder and extrapolation length is the mean free path in the surrounding medium. a is the radius of the cylinder, and α is the ratio of macroscopic absorption cross section in cylinder to scattering cross section in the surrounding medium. (Reproduced from S. A. Kushneriuk and C. McKay, *Neutron Density in an Infinite Non-Capturing Medium Surrounding a Long Cylindrical Body which Scatters and Captures Neutrons*, Atomic Energy of Canada Limited, CRT-566, July, 1954.)

$G(u, u')$ are the solutions of the steady-state

$$D \nabla^2 \phi - \Sigma_a \phi = 0$$

source-free diffusion equation in an infinite medium, the single source being considered a separate region of infinitesimal extent. The solutions apply (within the limitations of diffusion theory) throughout the infinite medium except infinitesimally near the source. Kernels for a number of source geometries are given in Table 1.

2.26 Solutions of the Steady-state Diffusion Equation in Finite Media with Localized Sources. If the geometry of the medium (and of any sources which may exist in the medium or at its boundaries) can be described by a single coordinate, the solutions are simple. Table 2 gives the general solutions for such one-dimensional cases in the usual geometries, as well as the solutions of some typical specific problems.

The three-dimensional case will be illustrated by a problem in rectangular coordi-

nary boundary conditions, is used in many reactor calculations when the absorption is not extremely large.

2.25 Solutions of the Steady-state Diffusion Equation in Infinite Medium—The Diffusion Kernels. The steady-state diffusion equation is

$$D \nabla^2 \phi - \Sigma_a \phi + S(r) = 0 \quad (16)$$

The equation is linear. Physically, this is because the mutual interactions between free neutrons are completely negligible at practical flux densities. Because of the linearity property, any superposition of solutions is a solution of the equation, and the flux distribution resulting from any arbitrary distribution of sources is just the superposition of the flux distributions due to the individual sources.

It is therefore useful to tabulate the flux distributions due to unit delta-function sources of various geometries (point sources, line sources, etc.) in an infinite medium. The flux distribution for such a unit source is called the kernel for the source. It will be denoted by $G(u, u')$. The quantity $G(u, u')$ is just the flux at u due to unit source at u' . The flux $\phi(u)$ at u due to an arbitrary distribution of sources $S(u')$, defined in the volume V in an infinite medium, is

$$\phi(u) = \int_V S(u') G(u, u') dV' \quad (17)$$

where the differential dV' signifies that the volume integral is to be taken over the source distribution. The kernels

Table 1. Diffusion Kernels in Infinite Media

Source geometry	Symbol for kernel	Source strength and location	Kernel = neutron flux at r (or at x , as the case may be)
Point	$G_{pt}(r, r')$	1 neutron/sec at r'	$\frac{e^{-\kappa r-r' }}{4\pi D r-r' }$
Plane	$G_{pl}(x, x')$	1 neutron/(cm ²)(sec) over a plane of infinite extent at x'	$\frac{e^{-\kappa x-x' }}{2\kappa D}$
Line of infinite length	$G_l(r, r')$	1 neutron/(cm)(sec) over a line of infinite length, in z direction, at r'	$\frac{1}{2\pi D} K_0(\kappa r-r')$ The vectors r and r' are both taken from the z axis, in a plane normal to the axis
Spherical shell	$G_s(r, r')$	1 neutron/sec per shell of radius r' , with center at origin	$\frac{1}{8\pi r r' \kappa D} (e^{-\kappa r-r' } - e^{-\kappa r+r' })$
Cylindrical shell of infinite length	$G_c(r, r')$	1 neutron/sec per shell of radius r' and unit length, with center at z axis	$\frac{1}{2\pi D} K_0(\kappa r) I_0(\kappa r') r > r'$ $\frac{1}{2\pi D} K_0(\kappa r') I_0(\kappa r) r < r'$

$|r - r'|$ denotes the absolute value of the vector difference of r and r' . I_0 and K_0 are the modified Bessel functions of the first and second kinds, respectively, of zero order.

notes, for which the diffusion equation in the source-free regions is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \kappa^2 \phi = 0 \tag{18}$$

If the medium consists of a single material, the equation may be assumed to be separable and to have a general solution of the form

$$\phi = X(x)Y(y)Z(z)$$

where X is a function of x alone, Y a function of y alone, etc. X , Y , and Z are then the solutions of the equations

$$\frac{d^2 X}{dx^2} + \alpha^2 X = 0 \quad \frac{d^2 Y}{dy^2} + \beta^2 Y = 0 \quad \frac{d^2 Z}{dz^2} + \gamma^2 Z = 0$$

where α , β , and γ are constants chosen to fit the boundary conditions plus the additional condition

$$\alpha^2 + \beta^2 + \gamma^2 + \kappa^2 = 0 \tag{19}$$

The rectangular prism is a typical problem, and is the arrangement frequently used for the measurement of diffusion length. If the prism is infinite in the z direction and is bounded in the x and y directions by the planes $x = 0, a$; $y = 0, b$; and if the source of effective strength Q neutrons/sec is located at the point (x', y', z') , the above procedure leads to the solution

$$\phi(x, y, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_{mn}(z) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{20}$$

where
$$\phi(z)_{mn} = \frac{2Q}{abD\gamma_{mn}} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y'}{b} \exp(-\gamma_{mn}|z - z'|)$$

and
$$\gamma_{mn}^2 = \kappa^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

Table 2. Solutions of Steady-state Diffusion Equation in Finite One-dimensional Media with Localized Sources
(For $\kappa > 0$)

Geometry of medium	Expanded form of diffusion equation	General solution	Typical specific problem	Solution of specific problem
Slab, infinite in y and z directions	$\frac{d^2\phi}{dx^2} - \kappa^2\phi = 0$	$\phi(x) = A \sinh \kappa x + B \cosh \kappa x$	Extrapolated slab boundaries at $x = -a$ and $x = +a$; infinite plane source of uniform strength Q neutrons/(cm ²)(sec) at $x = x'$ Net diffusion current of uniform density J neutrons/(cm ²)(sec) is flowing into slab at $x = 0$; slab extends to extrapolated boundary $x = a$	$\phi(x) = \frac{Q}{\kappa D} \frac{\sinh \kappa(a+x')}{\sinh 2\kappa a} \sinh \kappa(a-x)$ for $x > x'$ $\phi(x) = \frac{Q}{\kappa D} \frac{\sinh \kappa(a-x')}{\sinh 2\kappa a} \sinh \kappa(a+x)$ for $x < x'$ $\phi(x) = \frac{J}{\kappa D \cosh \kappa a} \sinh \kappa(a-x)$
Sphere	$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} - \kappa^2\phi = 0$	$\phi(r) = A \frac{\sinh \kappa r}{r} + B \frac{\cosh \kappa r}{r}$	Point source of strength Q neutrons/sec at center of sphere of extrapolated radius = a Spherical shell, of extrapolated outer radius = a , inner radius = b ; diffusion current of uniform density J neutrons/(cm ²)(sec) flowing into shell at $r = b$	$\phi(r) = \frac{Q}{4\pi D \sinh \kappa a} \frac{\sinh \kappa(a-r)}{r}$ $\phi(r) = \frac{J \left[\frac{\sinh \kappa(a-r)}{r} \right]}{D \left[\frac{\sinh \kappa(a-b)}{b^2} - \kappa \cosh \kappa(a-b) \right]}$
Infinitely long cylinder	$\frac{d^2\phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} - \kappa^2\phi = 0$	$\phi(r) = AI_0(\kappa r) + BK_0(\kappa r)$	Cylindrical shell, of extrapolated outer radius = a , inner radius = b ; diffusion current of uniform density J neutrons/(cm ²)(sec) flowing into shell at $r = b$	$\phi(r) = AI_0(\kappa r) + BK_0(\kappa r)$ where $A = \frac{J}{\kappa D} \frac{K_0(\kappa a)}{K_0(\kappa a)I_1(\kappa b) + I_0(\kappa a)K_1(\kappa b)}$ $B = \frac{J}{\kappa D} \frac{I_0(\kappa a)}{K_0(\kappa a)I_1(\kappa b) + I_0(\kappa a)K_1(\kappa b)}$

A and B are arbitrary constants, to be chosen to fit the boundary conditions.

D is the diffusion coefficient for the medium; $\kappa^2 = \Sigma_a/D = 1/L^2 = 1/\text{diffusion area for the medium}$.

I_0 and K_0 are the modified Bessel functions of first and second kinds, respectively, of zero order; I_1 and K_1 are the modified Bessel functions of first order.

For large values of $|z - z'|$ all terms of the summation except those for which $m = 1$, $n = 1$ become negligible, and the solution becomes

$$\phi(x, y, z) = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \exp \left[- \left(\kappa^2 + \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right)^{\frac{1}{2}} |z - z'| \right] \quad (21)$$

where
$$A = \frac{2Q}{abD[\kappa^2 + (\pi^2/a^2) + (\pi^2/b^2)]^{\frac{1}{2}}} \sin \frac{\pi x'}{a} \sin \frac{\pi y'}{b}$$

For this case the quantities π^2/a^2 and π^2/b^2 are called the *bucklings* in the x and y directions, respectively.

If the prism is finite in the z direction, bounded by the planes $z = 0, c$, the solution is again that given by Eq. (20), but with †

$$\begin{aligned} \phi(z)_{mn} &= \frac{4Q}{abD\gamma_{mn}} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y'}{b} \frac{\sinh \gamma_{mn} z'}{\sinh \gamma_{mn} c} \sinh \gamma_{mn}(c - z) & \text{for } z > z' \\ \phi(z)_{mn} &= \frac{4Q}{abD\gamma_{mn}} \sin \frac{m\pi x'}{a} \sin \frac{n\pi y'}{b} \frac{\sinh \gamma_{mn}(c - z)}{\sinh \gamma_{mn} c} \sinh \gamma_{mn} z & \text{for } z < z' \end{aligned} \quad (22)$$

and γ_{mn} as before.

2.3 The Albedo‡

If neutrons are incident on the boundary of a source-free medium, a large fraction of the neutrons which cross the boundary into the medium may, after one or more collisions, be scattered back across the boundary and out of the medium. The albedo β is a reflection coefficient which characterizes this process quantitatively. It is defined as the ratio of current density out of the medium (negative) to current density into the medium (positive):

$$\beta = \left[\frac{J-}{J+} \right]_{\text{boundary}} = \left[\frac{(\phi/4) + (D/2)(\partial\phi/\partial x)}{(\phi/4) - (D/2)(\partial\phi/\partial x)} \right]_{\text{boundary}} \quad (23)$$

where x is measured along a vector normal to the boundary and directed into the medium.

Within the limitations of diffusion theory, the albedo is characteristic only of the geometry and the material characteristics of the medium. Albedos for a number of geometries are given below.

2.31 Infinite Medium with Plane Boundary:

$$\beta = \frac{1 - 2\kappa D}{1 + 2\kappa D} \quad (24)$$

For the case of $\kappa D \ll 1$,

$$\beta \approx 1 - 4\kappa D \quad (25)$$

If the scattering is isotropic also, $\kappa D = (1/\sqrt{3}) \sqrt{\Sigma_a/\Sigma}$.

But Σ/Σ_a is just the total number of collisions (N) made by the average neutron in the medium before absorption, and

$$\beta = 1 - \frac{4}{\sqrt{3}} \frac{1}{\sqrt{N}} \quad (26)$$

This relationship aids in the physical understanding of the neutron "reflection" process.

Within the framework of diffusion theory only an isotropic distribution of velocities of incident neutrons can be considered. In many practical cases the distribution is

† The solution is valid regardless of which dimension of the prism is taken as the z dimension, i.e., regardless of the dimension in which the sinh term is used to describe the distribution. However, convergence of the series is good only if the longest dimension is taken as the z dimension.

‡ See Ref. 3. See also Art. 4.4 of Sec. 6-1.

not isotropic (e.g., a neutron beam incident on the medium). The albedo as derived from more detailed considerations of neutron transport is, for the case of low absorption and isotropic scattering,

$$\beta = 1 - \frac{1}{\sqrt{N}} \quad \text{for glancing incidence} \quad (27)$$

$$\beta = 1 - \frac{2.9078}{\sqrt{N}} \quad \text{for normal incidence} \quad (28)$$

The albedos below, for other geometries, apply for isotropic incidence only.

2.32 Slab of Finite Thickness a :

$$\beta = \frac{1 - 2\kappa D \coth \kappa a}{1 + 2\kappa D \coth \kappa a} \quad (29)$$

2.33 Sphere of Radius a :

$$\beta = \frac{1 - 2\kappa D[\coth \kappa a - (1/\kappa a)]}{1 + 2\kappa D[\coth \kappa a - (1/\kappa a)]} \quad (30)$$

2.34 Infinite Medium Surrounding Sphere of Radius a :

$$\beta = \frac{1 - 2D[\kappa + (1/a)]}{1 + 2D[\kappa + (1/a)]} \quad (31)$$

2.35 Spherical Shell, Inner Radius a , Outer Radius b ; Neutrons Incident from Inside:

$$\beta = \frac{1 - 2D[\kappa \coth \kappa(b-a) + (1/a)]}{1 + 2D[\kappa \coth \kappa(b-a) + (1/a)]} \quad (32)$$

3 SLOWING DOWN OF NEUTRONS

The kinetic energy of fission neutrons varies over a wide range, but it is in all cases high compared with thermal energies. In the course of their lives in a reactor such neutrons will, in general, lose energy by nonabsorbing collisions with nuclei of the reactor materials. The energy spectrum of neutrons in a typical thermal reactor will consist of a slowing-down spectrum which describes the distribution of energies among the neutrons being slowed down, plus a thermal spectrum of neutrons which have come into equilibrium with the thermal energy distribution of the reactor atoms. The slowing-down distribution will be described in this section, and the thermal distribution in Art. 4.

For very fast neutrons the slowing-down collisions may be either elastic or inelastic. The elastic process is by far the more important one in determining the effective behavior of neutrons over the entire energy spectrum in a thermal reactor and is the process which will be considered here. The effects of inelastic slowing down can, however, be included in the constants, such as τ (Art. 3.4), which describe the slowing-down distribution in the reactor.

3.1 Loss of Energy in a Single Elastic Collision

The loss of energy in a single collision with a stationary nucleus depends upon the mass number of the scattering nucleus and upon the angle θ between the original direction of the neutron velocity and the direction after the collision. It is useful in describing the mechanics of elastic collisions to employ the center of mass coordinate system. The origin for this system is at the center of mass of the neutron plus the nucleus with which it is colliding. Thus the center of mass (or C) system moves at constant velocity with respect to the laboratory (or L) coordinate system, and the collision always takes place at the origin of the C system. The scattering angle θ

in the L system is related to the scattering angle ϕ of the C system by

$$\cos \theta = \frac{1 + A \cos \phi}{\sqrt{1 + A^2 + 2A \cos \phi}} \quad (33)$$

where A is the mass number of the scattering nucleus.

The ratio of the neutron energy after collision to the energy before collision is

$$\frac{E}{E_0} = \frac{1+r}{2} + \left(\frac{1-r}{2}\right) \cos \phi \quad (34)$$

where

$$r = \left(\frac{A-1}{A+1}\right)^2 \quad (35)$$

The minimum value of E/E_0 occurs for $\phi = 180^\circ$ (head-on collision) and is equal to r . For neutrons in the energy ranges important for thermal reactors, the assumption is usually made that all values of $\cos \phi$ are equally probable (the scattering is spherically symmetric in the C system) and hence all values of E/E_0 from 1 to r are equally probable. The average logarithmic change in energy per collision ξ is then

$$\xi = -\ln \frac{\bar{E}}{E_0} = 1 + \frac{r}{1-r} \ln r \quad (36)$$

The limiting values of ξ for $A = 1$ (hydrogen) and $A = \infty$ are, respectively, 1 and 0.

If the moderator contains n different species of nuclides, of scattering cross section Σ_{si} and average logarithmic energy decrement per collision ξ_i , and if all the scattering cross sections are either independent of energy or vary in the same way with energy, then the average logarithmic energy decrement per collision $\bar{\xi}$ for the mixture is

$$\bar{\xi} = \frac{\sum_{i=1}^n \Sigma_{si} \xi_i}{\sum_{i=1}^n \Sigma_{si}} \quad (37)$$

Lethargy. Frequently, in treating the slowing down of neutrons it is convenient to employ a dimensionless variable, called the lethargy, in place of the energy variable. The lethargy u is defined by:

$$u = \ln \frac{E_0}{E}$$

where E_0 is the initial energy of the neutron and E is the energy at the instant in question. The quantity ξ , then, is just the average change in lethargy of a neutron per collision.

3.2 Collision and Slowing-down Densities

3.21 The slowing-down density q in a medium is defined as the number of neutrons that slow down past a given energy E per cubic centimeter per second. In the general case, q is a function of energy and position. In an infinite medium which absorbs only thermal neutrons and which is supplied by a uniform source of Q fast neutrons per cubic centimeter per second of energy E , the slowing-down density is constant and equal to Q at all energies from thermal to E .

3.22 The collision density per unit energy, $F(E)$, is simply the number of collisions made by neutrons with atoms of the medium per cubic centimeter per second per unit energy interval at the energy E . If the neutron flux per unit energy interval at energy E is $\phi(E)$ and the scattering cross section is $\Sigma_s(E)$, then the collision density is

$$F(E) = \Sigma_s(E) \phi(E) \quad (38)$$

In an infinite medium which absorbs only thermal neutrons, the collision density as a function of energy is given by

$$F(E) = \frac{Q}{E} \quad (39)$$

if the medium is hydrogen. For moderators other than hydrogen the collision density is

$$F(E) = \frac{Q}{\xi E} \quad (40)$$

for all energies E below about $r^3 E_0$ [see Eq. (35)]. For higher energies the variation of $F(E)$ is more complicated; it is discontinuous at the energy rE_0 because of the circumstance that above this energy neutrons are present which have suffered only one collision (as well as those that have had multiple collisions) whereas below rE_0 all neutrons have had at least two collisions.⁴

Equation (40) applies for a moderator composed of a mixture of atomic species provided it is possible to write an average logarithmic energy decrement ξ [see Eq. (37)] for the mixture. The neutron flux $\phi(E)$ can be obtained from Eq. (39) or (40) by use of Eq. (38).

$$\phi(E) = \frac{F(E)}{\Sigma_s(E)} = \frac{Q}{E} \frac{1}{\xi \Sigma_s(E)} \quad (41)$$

Thus, over any energy range for which Σ_s is constant, the flux is just proportional to $1/E$. This condition holds approximately for many moderators over the resonance energy region.

3.3 Spatial Distribution of Slowed Neutrons

If there is a point source of fast neutrons in a uniform moderating medium of infinite extent, the slowing-down density at any energy below the source energy will have some spatial distribution which is spherically symmetrical about the source and which is

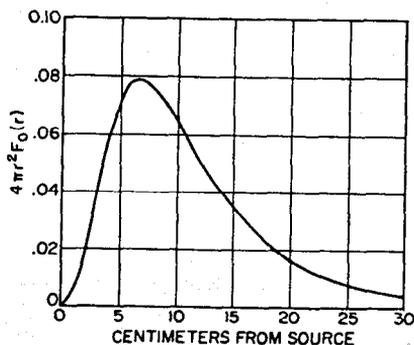


FIG. 3. Flux of 1.44 eV neutrons from a point source of 1 fission neutron per second in an infinite medium of H_2O . $F_0(r)$ is the flux in neutrons/(cm^2)(sec)(ev). (Reproduced from J. E. Wilkins, R. L. Hellens, and P. F. Zweifel, *Status of Experimental and Theoretical Information on Neutron Slowing Down Distributions in Hydrogenous Media*, Geneva Conf. Paper A/Conf.8/P/597, June 30, 1955.)

otherwise characteristic of the energy and of the moderator. If we are concerned with the calculation of a thermal reactor, we are interested in the slowing-down density at some energy just above thermal. Each fission in such a reactor is a source of fast neutrons. If we know the characteristic distribution of slowing-down density into the thermal-energy region from such a fast source, then we can treat each fission as a distributed source of thermal neutrons which diffuse according to the laws of Art. 2 until they produce further fissions to maintain the chain reaction. By the principle of *superposition* (see Art. 2.25), the effects of all the sources can be added up independently to give the reactor equation. The method by which this is done is treated in Art. 5. We consider here the methods of describing the slowing-down distribution.

Figure 3 is a plot of the measured slowing-down density distribution, at the indium resonance energy, from a point fission source in water, (the indium resonance energy is a convenient one at which to make the measurement; the distribution is nearly the same as that just above thermal). This curve is actually the kernel of the slowing-down density (see Art. 2.25) for the case in question. If such kernels are

to be incorporated into a reactor equation, it is evidently desirable to describe them by some relatively simple equation. Furthermore, it is desirable to be able to compute the distributions from microscopic data. It is by no means obvious that the distribution should be describable by a simple mathematical function; indeed, in the general case, it is not. The Fermi approximation, discussed below, gives a simple description which is satisfactory for many cases. For an extensive treatment of the slowing-down distribution by more rigorous methods, see Refs. 5 and 6.

3.4 The Fermi Age Approximation

This approximation is valid only if the scattering mean free path does not vary appreciably within any energy interval E to E' , where $E' = rE$ [see Eq. (35)], and if the number of collisions experienced by the neutron in reaching the energy of interest is large. The approximation is therefore quite poor for media in which the moderation is mainly by hydrogen or deuterium, but it is widely used for moderators of higher atomic weight than these. The approximation does not describe properly the slowing-down distribution at distances very far from the source, but at such distances the slowing-down density is too low to be important in most reactor problems.

3.41 The Fermi Age. If we have source neutrons produced at energy E_0 and are interested in the distribution as a function of E for all energies E less than E_0 , we define a new variable, the *Fermi age* τ , such that

$$\tau(E) = \int_E^{E_0} \frac{D(E) dE}{\xi \Sigma_s(E) E} \quad (42)$$

where $D(E)$ is the diffusion coefficient (see Art. 2.13), Σ_s is the macroscopic scattering cross section, and ξ is the logarithmic energy decrement per collision. The slowing-down density q then becomes a function of space and of τ , and its distribution is given by the Fermi age equation.

3.42 Fermi Age Equation:

$$\nabla^2 q(r, \tau) = \frac{\partial q(r, \tau)}{\partial \tau} \quad (43)$$

for all values of $\tau > 0$ (i.e., for all energies less than E_0). The initial condition is

$$\lim_{\tau \rightarrow 0} \nabla^2 q(r, \tau) = \frac{\partial q(r, \tau)}{\partial \tau} + S(r) \quad (44)$$

where $S(r)$ is the source strength of neutrons of $\tau = 0$ (i.e., energy E_0); ∇^2 is, of course, the Laplacian operator.

Note that τ , although referred to as the *age*, has the dimension (length)². It can be shown to be one-sixth the mean square (crow flight) distance traveled by a neutron from the time of its emission by the source until it reaches the energy E , which corresponds to the value of τ . An alternate statement is that τ is one-sixth the second spatial moment of the slowing-down density where the second spatial moment $\bar{r}^2(\tau)$ is defined by

$$\bar{r}^2(\tau) = \frac{\int_0^\infty r^2 q(r, \tau) 4\pi r^2 dr}{\int_0^\infty q(r, \tau) 4\pi r^2 dr} \quad (45)$$

$q(r, \tau)$ being the slowing-down density from a point source at the origin. Equation (42) can be used to compute τ for a *mixture* of elements provided the average value [Eq. (37)] is used for ξ .

3.43 Solutions of the Age Equation. The Slowing-down Kernels. The solution of the age equation for a unit point source of monoenergetic neutrons at r_0 in an infinite medium, the neutrons being emitted at an energy corresponding to $\tau = 0$, is

$$q(r, \tau) = \frac{1}{(4\pi\tau)^{3/2}} \exp\left(\frac{-|r - r_0|^2}{4\tau}\right) \quad (46)$$

This solution is the *point kernel* for the slowing-down density. It is a displacement kernel; the slowing-down kernels for a distribution of sources may be superimposed to give the net slowing-down density, as discussed for the diffusion kernels in Art. 2.25.

Note that the slowing-down density has the same spatial distribution as the Gauss error curve. In a given moderator containing a single point source, the slowing-down density is always highest near the source; it falls off more or less rapidly with distance according as τ is small (high energy) or large (low energy).

The kernels for the slowing-down density in an infinite medium with various source geometries are given in Table 3.

Table 3. Gaussian Slowing-down Kernels in Infinite Media

The kernel gives the slowing-down density, at point r (or x as the case may be) and age τ , produced by unit sources of neutrons of age τ' located at point r' (or x'). The flux $\phi(r, \tau)$ can be obtained by dividing the kernel by the slowing-down power $\xi\Sigma_s(r)$, provided the value of τ is one which corresponds to the slowing-down distribution (i.e., provided the neutrons are not thermalized).

Source geometry	Symbol for kernel	Source strength and location	Kernel
Point	$P(\tau, \tau; r', \tau')$	1 neutron/sec at r', τ'	$\frac{\exp \left[-\frac{ r - r' ^2}{4(\tau - \tau')} \right]}{[4\pi(\tau - \tau')]^{3/2}}$
Plane	$P_{pl}(x, \tau; x', \tau')$	1 neutron of age $\tau'/(cm^2)(sec)$ over a plane of infinite extent at x'	$\frac{\exp \left[-\frac{ x - x' ^2}{4(\tau - \tau')} \right]}{[4\pi(\tau - \tau')]^{1/2}}$
Line	$P_l(r, \phi, \tau; r', \phi', \tau')$	1 neutron of age $\tau'/(cm)(sec)$ over a line of infinite length, at r', ϕ'	$\frac{\exp \left[-\frac{\rho^2}{4(\tau - \tau')} \right]}{4\pi(\tau - \tau')}$ where $\rho^2 = r^2 + r'^2 - 2rr' \cos(\phi - \phi')$
Spherical shell	$P_s(r, \tau; r', \tau')$	1 neutron of age τ'/sec per shell of radius r' , with center at origin	$\frac{1}{4\pi r r'} \left\{ \frac{\exp \left[-\frac{ r - r' ^2}{4(\tau - \tau')} \right] - \exp \left[-\frac{ r + r' ^2}{4(\tau - \tau')} \right]}{4\pi(\tau - \tau')^{1/2}} \right\}$
Cylindrical shell	$P_c(r, \tau; r', \tau')$	1 neutron of age τ'/sec per cm length of infinitely long shell, of radius r' , with axis at $r = 0$	$\frac{\exp \left[-\frac{ r - r' }{4(\tau - \tau')} \right]}{4\pi(\tau - \tau')} I_0 \left(\frac{rr'}{2 \tau - \tau' } \right)$

3.44 Slowing-down Distribution from Fission Source. If the number of neutrons emitted per second between energy E' and $E' + dE'$ by a point source at the origin, in an infinite medium, is $F(E') dE'$, then the slowing-down density from the source is

$$q[\tau, \tau(E)] = \int_E^\infty \frac{e^{-\tau^2/4[\tau(E) - \tau(E')]} F(E') dE'}{\{4\pi[\tau(E) - \tau(E')]\}^{3/2}} \quad (47)$$

Similar equations will hold for the other slowing-down kernels.

Evidently, the slowing-down density at any energy E , as given by Eq. (47), is no longer Gaussian. Nevertheless, in reactor theory, it is often the practice to define an *effective age* $\bar{\tau}$, which can be used with the point kernel for a monoenergetic source [Eq. (45)], to describe the neutron balance in the reactor. The proper $\bar{\tau}$ is, however, a function of the size of the reactor (or, more precisely, of the buckling B^2). For very large reactors (small buckling) the effective value of τ for the slowing-down density just above thermal energy is given by

$$\bar{\tau} = \frac{1}{S} \int_{E_{th}}^\infty \tau(E', E_{th}) F(E') dE' \quad (48)$$

where S is the total source strength. In other words, for this case, one uses just the arithmetic mean of the ages of all the fission neutrons, from their source energy to thermal.

3.5 Slowing Down with Absorption

3.51 Resonance Escape Probability. In an infinite medium containing a uniform source of high-energy neutrons of strength $Q(E_0)$ neutrons/(cm^3)(sec), the slowing-down density $q(E)$ at some lower energy E will be just equal to $Q(E_0)$ if there is no absorption. If there is absorption, the slowing-down density at E will be some fraction $p(E)$ of the source strength. This fraction is called the *resonance escape probability*:

$$q(E) = p(E)Q(E_0) \quad (49)$$

For a homogeneous medium of total macroscopic scattering cross section $\Sigma_s(E)$, total absorption cross section $\Sigma_a(E)$, and effective logarithmic energy decrement $\xi(E)$, the resonance escape probability is calculated by

$$p(E) = \exp \left\{ - \int_E^{E_0} \frac{\Sigma_a(E')}{\xi(E')[\Sigma_s(E') + \Sigma_a(E')] E'} dE' \right\} \quad (50)$$

The expression is correct if all the moderation is by hydrogen. It is valid for moderators of mass number greater than 1 if all the absorption is by a single narrow resonance or by a number of narrow resonances provided in every case the energy of a resonance is higher than that of the next lower resonance by a ratio which is at least as great as $(1/r)^3$ or $(1/r)^4$ [Eq. (35)].

If the moderator is of mass number greater than 1, and if the absorption cross section and the scattering collision density vary only slowly with energy, the resonance escape probability is given approximately by⁷

$$p(E) = \exp \left(- \int_E^{E_0} \frac{\Sigma_a}{\xi \Sigma_s + \nu \Sigma_a} \frac{dE'}{E'} \right) \quad (51)$$

where

$$\nu = \frac{1 - r - r\epsilon - r\epsilon^2/2}{1 - r - r\epsilon} \quad (52)$$

and

$$\epsilon = \ln \frac{1}{r} \quad (53)$$

Although many cases arise in which Eqs. (49) and (50) do not apply rigorously, they are used extensively in practice and give reasonably accurate results for most cases, particularly if the resonance absorption is not large.

It should be pointed out that in those cases where the absorption is by resonances, the above expressions will give very poor results unless the absorber is distributed in a very nearly homogeneous manner. For resonance absorption in heterogeneous media see Art. 9.4.

3.52 The spatial distribution of slowing-down density from a localized source of fast neutrons in an infinite medium will, of course, be modified from that discussed in Arts. 3.3 and 3.4 if the medium absorbs neutrons. If the Fermi model is used, and if the medium absorbs only weakly, it can be shown that the spatial distribution of slowing-down density $q'(r, E)$ from any source distribution in the absorbing medium is related to the slowing-down distribution $q(r, E)$, which would result from the same source distribution if there were no absorption, by

$$q'(r, E) = p(E)q(r, E) \quad (54)$$

where $p(E)$ is the resonance escape probability discussed above. It is generally assumed in reactor calculations that the same result applies regardless of the slowing-down kernel used and, indeed, that the result applies not only for an infinite medium but for the finite reactor as well.

3.53 The Resonance Integral. If the microscopic absorption cross section of a material, as a function of energy, is denoted by $\sigma_a(E)$, then the resonance integral

for the material between the energies E and E_0 is

$$\text{Resonance integral} = \int_E^{E_0} \sigma_a(E') \frac{dE'}{E'} \quad (55)$$

Note that since for typical moderators the flux is proportional to $1/E$ in the resonance region [cf. Eq. (41)], the resonance integral is just the integral of the absorption cross section weighted by a factor which is proportional to the flux in such a moderator.

3.54 Use of Resonance Integral to Calculate Resonance Escape Probability. For the case in which the macroscopic absorption cross section is very much smaller than the macroscopic scattering cross section of the medium, the resonance escape probability becomes

$$p(E) = \exp \left(- \int_E^{E_0} \frac{\Sigma_a}{\xi \Sigma_s} \frac{dE'}{E'} \right) = \exp \left(- \int_E^{E_0} \frac{N_a \sigma_a}{\xi \Sigma_s} \frac{dE'}{E'} \right) \quad (56)$$

where N_a is the number of absorber atoms per cubic centimeter. Again, if the scattering cross section is constant over the range of important absorption, the resonance escape probability can be written

$$p(E) = \exp \left(- \frac{1}{\xi} \frac{N_a}{\Sigma_s} \int_E^{E_0} \sigma_a \frac{dE'}{E'} \right) = \exp \left[- \frac{1}{\xi} \frac{N_a}{\Sigma_s} (\text{resonance integral}) \right] \quad (57)$$

Note that the ratio N_a/Σ_s is the reciprocal of the macroscopic scattering cross section of the medium per atom of absorber in the medium.

Measured values of the resonance integral are usually quoted for the energy interval from fission (E_0) to the cadmium cutoff. The upper limit of the energy is usually unimportant as long as it is high, since absorption cross sections are low in the million-electron-volt range of energies.

3.55 The effective resonance integral is so defined that an equation similar to Eq. (57) will apply even if the inequality $\Sigma_a \ll \Sigma_s$ does not hold:

$$p(E) = \exp \left[- \frac{1}{\xi} \frac{N_a}{\Sigma_s} (\text{effective resonance integral}) \right] \quad (58)$$

and

$$\begin{aligned} \text{Effective resonance integral} &= \int_E^{E_0} \frac{\Sigma_s}{\Sigma_s + \Sigma_a} \sigma_a \frac{dE'}{E'} \\ &= \int_E^{E_0} \frac{1}{1 + (N_a \sigma_a / \Sigma_s)} \sigma_a \frac{dE'}{E'} \end{aligned} \quad (59)$$

Note that for a given absorber the effective resonance integral will depend only on the quantity Σ_s/N_a , the scattering cross section per absorber atom. Thus, a few measurements of the effective integral as a function of this ratio will define its variation. The curve of effective resonance integral vs. Σ_s/N_a will, of course, extrapolate to the true resonance integral for very large values of Σ_s/N_a .

4 ENERGY DISTRIBUTION OF THERMAL NEUTRONS

Neutrons slowing down in a moderating medium, if they are not captured first, will eventually reach such low energies that upon collision with nuclei of the medium they may either lose energy or gain energy from the thermal motions of the nuclei. Ultimately they come into equilibrium with the nuclei of the medium, gaining, on the average, as much energy as they lose. The neutrons in such a case are referred to as thermal neutrons.

4.1 The Maxwell-Boltzmann Distribution

If the absorption is quite low, the neutron energy distribution follows the *Maxwell-Boltzmann law*, the same as that which describes the distribution of energy of thermal agitation among the nuclei of the moderator. If $n(E)$ is the number of neutrons per

unit volume per unit energy interval at the energy E , if n is the total number of thermal neutrons per unit volume, and if T is the absolute temperature of the moderator, the Maxwell-Boltzmann distribution of neutron energies is

$$\frac{n(E)}{n} = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT} \quad (60)$$

where k is the Boltzmann constant and is equal to 8.61×10^{-5} ev/°K.

The velocity distribution of the neutrons is given, in terms of $n(v)$, the number of neutrons per unit volume per unit energy interval, by

$$\frac{n(v)}{n} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT} \right)^{3/2} v^2 \exp \left(-\frac{mv^2}{2kT} \right) \quad (61)$$

where m is the mass of the neutron.

The most probable velocity α of the neutrons in a Maxwell-Boltzmann distribution is

$$\alpha = \sqrt{\frac{2kT}{m}} \quad (62)$$

The energy corresponding to this most probable velocity is

$$E = \frac{1}{2} m \alpha^2 = kT \quad (63)$$

but the most probable energy is $\frac{1}{2} kT$.

4.2 Specification of Neutron Energy: kT Neutrons

The energy of a thermal-neutron distribution is ordinarily characterized by giving the energy kT corresponding to the most probable velocity in the distribution. Neutrons in a thermal distribution of temperature T are sometimes referred to as kT neutrons. "Thermal cross sections" are usually quoted as the cross sections for monoenergetic neutrons of velocity 2,200 m/sec. This velocity corresponds to the kT value 0.0253 ev and is the most probable velocity for the temperature 20°C. For computing the number of processes occurring in a medium containing a thermal distribution of neutrons, an average cross section is usually derived, as discussed in Art. 4.3.

4.3 Averaging of Thermal Cross Sections

In a diffusion medium, if any process is characterized by an energy-dependent cross section $\Sigma(E)$, then the number of such processes occurring per unit volume per second per unit energy interval is $\phi(E)\Sigma(E)$, where $\phi(E)$ is the neutron flux per unit energy interval. In the thermal-energy region, the cross section may be averaged over the Maxwell-Boltzmann distribution of flux to obtain an average cross section which, when multiplied by the total thermal flux, gives the total number of processes occurring per unit volume per second in the thermal-energy region:

$$\begin{aligned} \bar{\Sigma} &= \frac{\int_0^\infty \Sigma(E) \phi(E) dE}{\int_0^\infty \phi(E) dE} = \frac{\int_0^\infty \Sigma(E) n(E) v(E) dE}{\int_0^\infty n(E) v(E) dE} \\ &= \frac{\int_0^\infty \Sigma(E) \left[\frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT} \right] \left(\frac{2E}{m} \right)^{1/2} dE}{\int_0^\infty \left[\frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT} \right] \left(\frac{2E}{m} \right)^{1/2} dE} \\ &= \frac{1}{(kT)^2} \int_0^\infty \Sigma(E) E e^{-E/kT} dE \quad (64) \end{aligned}$$

The averaging can be done numerically for any measured variation of the cross section. Obviously, either the macroscopic cross section $\Sigma(E)$ or microscopic cross section $\sigma(E)$ can be averaged.

The use of such averaged cross sections makes it feasible to apply the laws of diffusion of monoenergetic neutrons (Art. 2) to the thermal distribution of neutrons. Usually it is necessary to average only absorption or fission cross sections. In media which show strong variations of scattering cross section with energy in the thermal range (e.g., H₂O) the variations result from interactions between the neutron and the molecular structure of the medium or from interactions between the neutron and the crystal structure of the medium. In these cases the diffusion coefficient cannot be derived from a simple average of the scattering cross section, and the only reliable values are obtained experimentally.

If the cross section to be averaged is proportional to the reciprocal of the neutron velocity in the thermal region ($1/v$ variation), the relation between the average cross section $\bar{\sigma}$ and the kT cross section σ_{kT} is

$$\bar{\sigma} = \frac{\sqrt{\pi}}{2} \sigma_{kT} = \frac{\sigma_{kT}}{1.128} \quad (65)$$

and the average cross section for the absolute temperature T is related to the average cross section for the absolute temperature T_0 by

$$\frac{\bar{\sigma}(T)}{\bar{\sigma}(T_0)} = \sqrt{\frac{T_0}{T}} \quad (66)$$

For certain important cross sections which do not follow the $1/v$ variation, the averaging indicated by Eq. (64) has already been done over a range of values of kT and factors have been tabulated which correct for the departure from $1/v$ dependence. Thus, in the cross-sectional compilation of Hughes and Harvey⁸ a correction factor f is given as a function of energy for a number of nuclides. If the kT cross section of any of these nuclides, at energy E , is multiplied by the value of f corresponding to the same energy E , an effective kT cross section, $\sigma_{eff}(kT)$, is obtained. The use of this effective kT cross section in Eq. (65) will yield the correct value of $\bar{\sigma}$ averaged over the Maxwellian distribution which has its most probable velocity at the energy E .

4.4 Deviations from the Maxwell-Boltzmann Distribution

If absorption in the thermal region is large, the energy distribution of neutrons is modified by the resultant loss of neutrons. The effect for practically occurring absorbers is to shift the energy distribution toward higher energy values. The energy distribution in a medium of atomic hydrogen (constant scattering cross section) with $1/v$ absorption has been investigated by Wigner and Wilkins.^{9,10} These studies indicate that the absorption does not change the shape of the distribution drastically but shifts it toward higher energies. Experiments on existing reactors indicate that the effective neutron temperatures exceed the moderator temperatures by the order of 50°C.

Even in the case of weak absorption the Maxwell-Boltzmann equation cannot give the true energy distribution of neutrons in the energy range several times kT , for in this region the thermal distribution must merge into the $1/E$ distribution which characterizes the slowing-down process (Art. 3). These two regions may be joined approximately by equating the slowing-down density just above thermal energy to the total absorption of thermal neutrons

$$\phi(\text{thermal}) \bar{\Sigma}_a = q(E_{th}) = \phi(E) E \xi \Sigma_s(E)$$

where $\phi(E)$ is the flux per unit energy interval in the energy range just above thermal and is thus given in terms of the total thermal flux by

$$\frac{\phi(E)}{\phi(\text{thermal})} = \frac{\bar{\Sigma}_a}{E \xi \Sigma_s(E)} \quad (67)$$

On the other hand, $\phi(E)$, the flux per unit energy interval, is given in the thermal region in terms of ϕ (thermal) by the Maxwell-Boltzmann distribution

$$\begin{aligned} \frac{\phi(E)}{\phi(\text{thermal})} &= \frac{2}{\sqrt{\pi}} \frac{v(E)}{\bar{v}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT} \\ &= \frac{E}{(kT)^2} e^{-E/kT} \end{aligned} \quad (68)$$

The boundary between the thermal and the slowing-down energy regions can arbitrarily be defined for a given medium as the energy at which the two distributions (67) and (68) give the same value for $\phi(E)$. It is to be understood that this is a purely formal joining of the two distributions and that neither describes the true neutron energy distribution well in the transition region. In most practical problems a precise knowledge of the energy distribution in this region is not required.

5 CRITICAL REACTOR EQUATIONS FOR THERMAL AND NEAR-THERMAL REACTORS

The thermal-neutron diffusion equation expresses the balance between production and loss of thermal neutrons in a diffusion medium. In a thermal or near-thermal reactor the absorption of thermal neutrons by fissionable material leads to fissions which provide further neutrons, which slow down and ultimately become thermal. When the characteristics of the reactor materials and their geometry are so adjusted that the production of thermal neutrons by fission followed by slowing down (during which process some of the epithermal neutrons may leak from the reactor) is just equal to the loss of thermal neutrons by absorption and leakage, the neutron population of the reactor will remain constant in time and the reactor is said to be *critical*. The equation which satisfies this condition of criticality can be derived from the diffusion equation [Eq. (12)] by setting the rate of change of neutron density to zero and relating the source strength $S(\mathbf{r})$ of thermal neutrons to the fission process. For the latter purpose the *multiplication factor* k will be defined. In defining k , the medium (reactor) is imagined to be infinite in extent but to contain the same energy distribution of neutrons as the finite medium under consideration. The quantity k is defined as the number of thermal neutrons which would be produced in such an infinite medium, by fission followed by slowing down, per thermal neutron absorbed.

When the procedure described above is applied, a general reactor equation is derived, the specific form of which depends upon the neutron slowing-down law appropriate for the reactor composition. By the application of approximations which are valid for most practical cases, the formal solution of the general equation can be written in terms of the *wave equation*, which specifies the spatial distribution of neutrons, plus a *characteristic equation*, which specifies the conditions for criticality in terms of the *buckling* of the wave equation and the constants of the reactor materials.

The remainder of this section will present the reactor equation in detail, its general solution, and the form of the equation and its solutions for various common slowing-down laws. Detailed solutions for various geometries will be treated in Arts. 6, 7, and 8.

5.1 The Finite, Steady-state, Reactor Equation

If we apply the "monoenergetic" diffusion equation [Eq. (12)] to the thermal group of neutrons, using appropriately averaged values for the thermal diffusion properties of the medium (Art. 4) and identifying the local source strength of thermal neutrons $S(\mathbf{r})$ with the local value of the thermal slowing-down density $q(E_s, \mathbf{r})$ we have

$$D \nabla^2 \phi_s(\mathbf{r}) - \Sigma_a \phi_s(\mathbf{r}) + q(E_s, \mathbf{r}) = \frac{\partial n_s(\mathbf{r})}{\partial t} \quad (69)$$

where the subscript s designates the thermal (i.e., slow) group of neutrons.

In a chain-reacting system, the slowing-down density is that resulting from the slowing down of fission neutrons. We now designate by $P(E_s, r, r')$ the finite thermal slowing-down kernel, i.e., the probability that a fission neutron created at r' , inside the reactor, will become a thermal neutron in unit volume at r . The slowing-down density then becomes†

$$q(E_s, r) = \frac{k}{p} \int_{\text{reactor volume}} \Sigma_a \phi_s(r') P(E_s, r, r') dr' \quad (70)$$

where the integral is taken over the entire reactor volume. The finite reactor equation then becomes

$$D \nabla^2 \phi_s(r) - \Sigma_a \phi_s(r) + \frac{k}{p} \int_{\text{reactor volume}} \Sigma_a \phi_s(r') P(E_s, r, r') dr' = \frac{\partial n_s(r)}{\partial t} \quad (71)$$

or, substituting L^2 for D/Σ_a and applying the equation to a critical reactor ($\partial n_s/\partial t = 0$),

$$L^2 \nabla^2 \phi_s(r) - \phi_s(r) + \frac{k}{p} \int_{\text{reactor volume}} \phi_s(r') P(E_s, r, r') dr' = 0 \quad (72)$$

This is the finite reactor equation. Evaluation of the final term is evidently quite difficult, since $P(E_s, r, r')$ will depend upon the leakage probability of fission neutrons formed at r' . In the following section approximations are made which simplify the evaluation of $P(E_s, r, r')$.

5.2 The Asymptotic Reactor Equation

Far from reactor boundaries the finite slowing-down kernel can evidently be replaced by the corresponding infinite kernel $P_\infty(E_s, |r - r'|)$, which has the characteristic that its value is determined only by the absolute distance $|r - r'|$ between the point of measurement and the point at which the fission neutrons were born. Such a kernel is called a *displacement kernel*. The reactor equation for this condition becomes

$$L^2 \nabla^2 \phi_s(r) - \phi_s(r) + \frac{k}{p} \int_{\text{all space}} \phi_s(r') P_\infty(E_s, |r - r'|) dr' = 0 \quad (73)$$

It can be shown that if the extrapolation distance can be considered independent of neutron energy, and if the finite and infinite slowing-down kernels satisfy the same linear equation (true for the Fermi, group, and transport kernels and convolutions of them), then the asymptotic solution holds everywhere in a critical reactor, except within a distance of the order of a mean free path from the boundary, and the slowing-down density in the reactor is identical with the infinite slowing-down density except within a distance of the order of a mean free path from the boundary. The asymptotic equation can, therefore, be used for most practical reactor problems. It is this equation for which solutions will be given in the following sections.

5.3 General Solution of the Asymptotic Equation

The solutions of Eq. (73) are also solutions of the *wave equation*:

$$\nabla^2 \phi_s + B^2 \phi_s = 0 \quad (74)$$

where B^2 is a constant, which is generally referred to as the *buckling* of the reactor. Until further restrictions are put upon B^2 , the solutions of Eq. (74) specify only a family of possible spatial distributions of neutron flux in the reactor. It is necessary to consider further restrictions on the solution which take into account the neutron

† Historically, both k and $P(E_s, r, r')$ have been so defined as to take into account resonance absorption. The same definitions have been followed here. Hence the appearance of the resonance escape probability p in the denominator of Eq. (70).

balance within the reactor. Evidently, once a formal specification of the flux distribution has been made, as by Eq. (74), a formal evaluation can be made of the space integral of the slowing-down density [last term of Eq. (73)]. In practice this is done by taking the three-dimensional Fourier transform $\bar{P}_\infty(E_s, B^2)$ of the slowing-down density with respect to B^2 :†

$$\bar{P}_\infty(E_s, B^2) = \int_{-\infty}^{\infty} e^{iB(r-r')} P_\infty(E_s, |r-r'|) d(|r'-r|) \quad (75)$$

The characteristic equation of the transformed reactor equation is then

$$-L^2 B^2 - 1 + \frac{k}{p} \bar{P}_\infty(E_s, B^2) = 0 \quad (76)$$

and the complete statement of the general solution is as follows: The asymptotic reactor equation [Eq. (73)] is satisfied by any solution of the wave equation [Eq. (74)] provided B^2 is a root of the characteristic equation [Eq. (76)]. Furthermore, the slowing-down density $q_\infty(E, r)$ at any energy E is given by

$$q_\infty(E, r) = \frac{k}{p} \Sigma_a \bar{P}_\infty(E, B^2) \phi_s(r) \quad (77)$$

Equations (74) and (76) give two specifications for the buckling B^2 of the reactor. Equation (76) specifies B^2 in terms of the properties of the reactor materials. The buckling when so specified is sometimes referred to as the *material buckling* B_m^2 . On the other hand Eq. (74), when solved with appropriate boundary conditions, specifies B^2 in terms of the geometry of the reactor. When so specified, B^2 is often referred to as the *geometrical buckling* B_g^2 . Only if the reactor geometry and the characteristics of the reactor materials are such that $B_m^2 = B_g^2$ will there be a solution for the steady-state reactor equation that satisfies the boundary conditions. This condition ($B_m^2 = B_g^2$) is referred to as the *critical condition*.

The above treatment of the general reactor equation is obviously little more than a conceptual one. More detailed treatments will be helpful in fundamental applications of the theory.‡ However, for the practical purpose of calculating critical reactors for which the usual slowing-down distributions apply, values of $\bar{P}_\infty(E_s, B^2)$ have been tabulated (see following sections). For these cases the appropriate value of $\bar{P}_\infty(E_s, B^2)$ can be substituted into Eq. (76), and solution of the reactor equation involves only the finding of a consistent value of B^2 which satisfies both the algebraic equation (76) and the partial differential equation (74).

† If $\psi(x, y, z)$ is a function (scalar or vector) in x, y, z space, its Fourier transform $\bar{\psi}(\xi, \eta, \zeta)$ in the new ξ, η, ζ space is defined by the equation

$$\bar{\psi}(\xi, \eta, \zeta) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{i(x\xi + y\eta + z\zeta)} \psi(x, y, z)$$

A compressed notation is generally used, in which $(x, y, z) \approx r$; $(\xi, \eta, \zeta) \approx \omega$; $dx dy dz \approx dr$; $d\xi d\eta d\zeta \approx d\omega$. In this notation the transform and its inverse are

$$\begin{aligned} \bar{\psi}(\omega) &= \int \psi(r) e^{i\omega \cdot r} dr \\ \psi(r) &= \frac{1}{(2\pi)^3} \int \bar{\psi}(\omega) e^{-i\omega \cdot r} d\omega \end{aligned}$$

$\psi(r)$ and $\bar{\psi}(\omega)$ are either both scalars or both vectors.

If the function $\psi(r)$ depends only on $r = |r|$, then its transform $\bar{\psi}(\omega)$ depends only on $\omega = |\omega|$, and the transformations simplify to

$$\begin{aligned} \bar{\psi}(\omega) &= \frac{4\pi}{\omega} \int_0^\infty r \psi(r) \sin \omega r dr \\ \psi(r) &= \frac{1}{2\pi^2 r} \int_0^\infty \omega \bar{\psi}(\omega) \sin \omega r dr \end{aligned}$$

‡ The treatment outlined here is that developed by Weinberg and described in Ref. 2, vol. 2, part 1, chap. 5.

5.4 Gaussian Slowing-down Distribution (Fermi Age Theory)†

The infinite slowing-down kernel and the resulting characteristic equations are

$$\text{Kernel: } P_{\infty}(E_s, \mathbf{r}, \mathbf{r}') = \frac{p \exp[-|\mathbf{r} - \mathbf{r}'|^2/4\tau(E_s)]}{[4\pi\tau(E_s)]^{3/2}} \quad (78)$$

$$\text{Fourier transform: } \bar{P}_{\infty}(E_s, B^2) = p e^{-B^2\tau(E_s)} \quad (79)$$

$$\text{Characteristic equation: } 1 + L^2 B^2 = k e^{-B^2\tau(E_s)} \quad (80)$$

In these equations, $\tau(E_s)$ is the Fermi age from some "effective" value of the fission energy to thermal. Rigorously, this effective value depends upon B_s^2 , but the value for a medium of infinite extent can be used in most practical cases (see Art. 3.44).

It will be recalled that the Gaussian slowing-down distribution results from the assumption of "continuous" slowing down (Art. 3). The corresponding characteristic equation (80) therefore is used most successfully for moderators which approximate this condition, i.e., for relatively heavy atoms such as carbon.

5.5 Zero Slowing-down Length (One-group Theory)

In this representation it is assumed that the fission neutrons are born as thermal neutrons. It is a poor approximation for most cases; the "modified" one-group theory (see below) is a much better approximation of equal simplicity.

$$\text{Kernel: } P_{\infty}(E_s, \mathbf{r}, \mathbf{r}') = p \delta(\mathbf{r} - \mathbf{r}') \quad (81)$$

$$\text{Fourier transform: } \bar{P}_{\infty}(E_s, B^2) = p \quad (82)$$

$$\text{Characteristic equation: } 1 + L^2 B^2 = k \quad (83)$$

5.6 Exponential Slowing-down Distribution (Two-group Theory)

It is assumed that a fission neutron diffuses without change of energy and with a diffusion coefficient D_f until a single (fictitious) slowing-down event occurs which reduces its energy to thermal in one jump. A cross section Σ_{af} is assigned to specify the probability of the fictitious event per centimeter of travel, and a characteristic slowing-down length L_f is defined in analogy to the thermal diffusion length L :

$$L_f^2 = \frac{D_f}{\Sigma_{af}} = \frac{1}{\kappa_f^2} \quad (84)$$

The diffusion kernels of Art. 2 apply to this picture (when multiplied by Σ_{af} to give slowing-down density):

$$\text{Kernel: } P_{\infty}(E_s, \mathbf{r}, \mathbf{r}') = \frac{p \exp(-1/L_f)|\mathbf{r} - \mathbf{r}'|}{4\pi L_f^2 |\mathbf{r} - \mathbf{r}'|} \quad (85)$$

$$\text{Fourier transform: } \bar{P}_{\infty}(E_s, B^2) = \frac{p}{1 + L_f^2 B^2} \quad (86)$$

$$\text{Characteristic equation: } (1 + L^2 B^2)(1 + L_f^2 B^2) = k \quad (87)$$

For the most useful applications of the two-group model, it is convenient to write the two coupled partial differential equations which, for this model, are equivalent to the asymptotic reactor equation (73):

$$D_f \nabla^2 \phi_f - \Sigma_{af} \phi_f + \frac{k}{p} \Sigma_{as} \phi_s = 0 \quad (88)$$

$$D_s \nabla^2 \phi_s - \Sigma_{as} \phi_s + p \Sigma_{af} \phi_f = 0 \quad (89)$$

† See Art. 3.

‡ δ is the Dirac delta function, defined as a function whose value is zero at all values of x except $x = 0$ and whose value at $x = 0$ is such that $\int_{-\infty}^{\infty} \delta(x) dx = 1$.

In these equations the subscript s designates the thermal group of neutrons and the subscript f designates the fast group.

These two equations are simply the diffusion equations for the thermal flux ϕ_s and the fast flux ϕ_f . They are both satisfied by solutions of the wave equation

$$\nabla^2 \phi + B^2 \phi = 0$$

provided the solutions for ϕ_f and ϕ_s are so chosen as to preserve the relations between the fast and thermal fluxes as specified by Eq. (88) or (89); e.g.,

$$\frac{\phi_f}{\phi_s} = \frac{\Sigma_{as} + D_s B^2}{p \Sigma_{af}} \quad (90)$$

(see Art. 7).

Computation of the mean square slowing-down distance from the kernel [Eq. (85)] shows that it is just $6L_f^2$. Hence L_f^2 has the same significance in the two-group representation as the age τ has in the Fermi age representation. Obviously, the two characteristic equations (80) and (81) will not predict the same criticality conditions for a given reactor if identical values are used for τ and L_f^2 . Experience has shown that the two-group picture is the better approximation for hydrogen-moderated reactors and that reasonable agreement between experimental and calculated critical sizes for such reactors results if the measured value of the "age" or slowing-down area is used as the value for L_f^2 . For reactors moderated by relatively heavy materials such as graphite, the Fermi age picture is much the better approximation, and good results can be obtained in two-group calculations only if some adjustment is made on the value of L_f^2 .

Despite its limitations the two-group approximation is used extensively for multi-region calculations (e.g., reflected reactors) on many types of reactors because it is the simplest approximation which will give reasonably accurate results for such problems. When moderators other than those rich in hydrogen are involved, L_f^2 is often considered as an adjustable quantity whose magnitude can be adjusted to compensate for the relatively poor description which the diffusion kernel gives of the slowing-down distribution. For example, if a reflected reactor is to be solved, a guess may first be made of the equivalent bare dimensions of the reactor (Art. 7.6), and the equivalent bare reactor may be solved by some approximation better than the two group. An adjustment may then be made in L_f^2 to force the two-group calculation to give the same result for the equivalent bare reactor, and this adjusted L_f^2 may be used in the calculation of the reflected reactor.

5.7 Modified One-group Theory

If the left-hand side of Eq. (87) is expanded and the term in B^4 is neglected, we obtain the characteristic equation for the modified one-group approximation:

$$1 + (L^2 + L_f^2)B^2 = 1 + M^2 B^2 = k \quad (91)$$

A similar result is obtained if the Fermi age equation (70) is expanded to terms of first order in B^2 . The quantity $M^2 = L^2 + L_f^2$ (or $L^2 + \tau$) is called the *migration area*.

5.8 The Multigroup Approximation

In the multigroup representation treated here, the epithermal energy range is divided into a number of smaller energy intervals and all the epithermal neutrons are sorted into the same number of "groups" according to the energy interval into which they fall at the instant considered. A neutron of a given group is considered to diffuse at constant energy until a slowing-down event occurs which slows it into the next lower group in one jump. Every neutron, during its lifetime, is considered to pass through every energy interval (for a more general multigroup representation, see Sec. 10). Evidently the two-group representation is the special case of the n -group

representation in which $n = 2$. For the n -group case we define $n - 1$ fictitious slowing-down cross sections Σ_{ai} which describe the probability of slowing from one group to the next and $(n - 1) L_i$'s and κ_i 's which are related to the Σ_{ai} and the average diffusion constant D_i for each group:

$$L_i^2 = \frac{1}{\kappa_i^2} = \frac{D_i}{\Sigma_{ai}} \quad (92)$$

If we let $i = 1$ refer to the highest energy group, the slowing-down kernel into the n th (thermal) group is

$$P_\infty(E_s, \mathbf{r}_n, \mathbf{r}_1) = p \int_{-\infty}^{\infty} d\mathbf{r}_2 \int_{-\infty}^{\infty} d\mathbf{r}_3 \cdots \int_{-\infty}^{\infty} d\mathbf{r}_{n-1} \frac{\exp - \kappa_1 |\mathbf{r}_2 - \mathbf{r}_1|}{4\pi L_1^2 |\mathbf{r}_2 - \mathbf{r}_1|} \\ \times \frac{\exp - \kappa_2 |\mathbf{r}_3 - \mathbf{r}_2|}{4\pi L_2^2 |\mathbf{r}_3 - \mathbf{r}_2|} \cdots \frac{\exp - \kappa_{n-1} |\mathbf{r}_n - \mathbf{r}_{n-1}|}{4\pi L_{n-1}^2 |\mathbf{r}_n - \mathbf{r}_{n-1}|} \quad (93)$$

where $\kappa_i = 1/L_i$.

The Fourier transform of the kernel is

$$\bar{P}_\infty(E_s, B^2) = \frac{p}{(1 + L_1^2 B^2)(1 + L_2^2 B^2) \cdots (1 + L_{n-1}^2 B^2)}$$

and the characteristic equation is

$$\prod_{i=1}^n (1 + L_i^2 B^2) = k$$

where, it will be remembered, L_n^2 is just the thermal diffusion area L^2 .

The characteristic equation is thus an algebraic equation of degree n . If $k > 1$, there are always $n - 1$ complex or negative roots in addition to the one real, positive value for B^2 . In the case of a bare reactor, the complex and negative roots are eliminated by the boundary conditions.

5.9 Convolution of Slowing-down Kernels; Synthetic Kernels

Equation (93) is an example of a *convolution* of slowing-down kernels. If a kernel has the general form

$$P(|\mathbf{r}_n - \mathbf{r}_1|) = \int d\mathbf{r}_2 \int d\mathbf{r}_3 \cdots \int d\mathbf{r}_{n-1} [P_1(|\mathbf{r}_2 - \mathbf{r}_1|)] [P_2(|\mathbf{r}_3 - \mathbf{r}_2|)] \\ \times [P_3(|\mathbf{r}_4 - \mathbf{r}_3|)] \cdots [P_{n-1}(|\mathbf{r}_n - \mathbf{r}_{n-1}|)] \quad (94)$$

it is said to be a *convolution* of the kernels P_1, P_2, \dots, P_{n-1} . If the Fourier transforms of these kernels are, respectively, $\bar{P}_1(B^2), \bar{P}_2(B^2), \dots, \bar{P}_{n-1}(B^2)$, then the

Table 4. Kernels and Their Transforms

(L^2 and τ have the same meaning as in the discussion above; λ is the mean free path)

Type	Kernel $P_\infty(E, \mathbf{r}, \mathbf{r}')$	Transform $\bar{P}_\infty(E, B^2)$
Gaussian (Fermi)	$\frac{e^{- \mathbf{r}-\mathbf{r}' /4\tau}}{(4\pi\tau)^{3/2}}$	$e^{-B^2\tau}$
Diffusion	$\frac{e^{- \mathbf{r}-\mathbf{r}' /L}}{4\pi L^2 \mathbf{r}-\mathbf{r}' }$	$\frac{1}{1 + L^2 B^2}$
Transport	$\frac{e^{- \mathbf{r}-\mathbf{r}' /\lambda}}{4\pi\lambda \mathbf{r}-\mathbf{r}' ^2}$	$\frac{1}{\lambda B} \tan^{-1}(\lambda B)$

transform of the convolution kernel P is simply the product

$$\bar{P}(B^2) = [\bar{P}_1(B^2)][\bar{P}_2(B^2)] \cdots [\bar{P}_{n-1}(B^2)] \quad (95)$$

Convolutions of kernels of the same or different types may be used to approximate the slowing-down distributions in various moderators. The three types of kernels which have been employed, with their transforms, are listed in Table 4.

If a convolution kernel is arbitrarily constructed to fit the measured slowing-down distribution in a moderator without regard to the physical aspects of the slowing-down process, the result is often called a *synthetic kernel*. For example, the slowing-down distribution in H_2O has been described by a convolution of four diffusion kernels whose characteristic lengths $[L_i, \text{Eq. (92)}]$ were arbitrarily chosen to fit the experimental measurements.

6 SOLUTIONS OF THE WAVE EQUATION FOR HOMOGENEOUS BARE REACTORS

As pointed out in Art. 5, solution of the critical reactor equation consists of finding a value of B^2 which satisfies both the wave equation (74) and the characteristic equation (76). We consider here the solutions of the wave equation for homogeneous bare reactors. A one-dimensional (slab) and a three-dimensional (rectangular parallelepiped) case will be treated as examples.

6.1 Infinite Slab

For a chain-reacting slab, infinite in the y and z directions, the wave equation becomes

$$\frac{d^2\phi}{dx^2} + B^2\phi = 0$$

If we take the center plane of the slab as $x = 0$, the solution is

$$\phi = A \cos Bx + C \sin Bx$$

where A and C are constants, to be chosen to fit the boundary conditions. Since ϕ must be symmetrical about $x = 0$, the constant C must be zero, and the solution

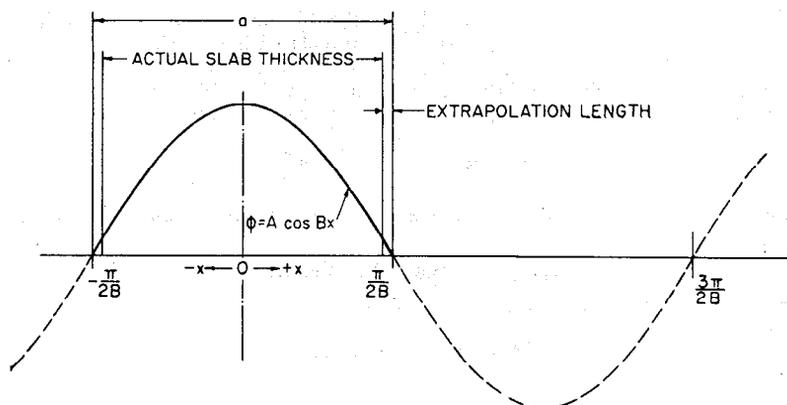


FIG. 4. Solution of the wave equation for infinite bare slab.

contains only the cosine term (Fig. 4). The boundary condition is $\phi = 0$ on the extrapolated boundary of the slab. This condition is met if the half thickness of the slab ($a/2$) is equal to $\pi/2B$ or $3\pi/2B$ or $5\pi/2B$. . . , etc.

Table 5. Solutions of the Wave Equation for Homogeneous Bare Reactors

Reactor shape	Extrapolated boundaries	Form of $\nabla^2\phi + B^2\phi = 0$	Solution	Critical dimensions
Slab; infinite in y and z direction, thickness = a in x direction; central plane at $x = 0$	Planes $x = \pm \frac{a}{2}$	$\frac{d^2\phi}{dx^2} + B^2\phi = 0$	$\phi = A \cos \frac{\pi x}{a}$ $B^2 = \frac{\pi^2}{a^2}$	$a = \frac{\pi}{B}$
Infinitely long cylinder of radius R	$r = R$	$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + B^2\phi = 0$	$\phi = A J_0 \left(\frac{2.405}{R} r \right)$ $B^2 = \frac{(2.405)^2}{R^2}$	$R = \frac{2.405}{B}$
Cylinder of radius R , length H ; origin of coordinates at center	$r = R$ Planes $z = \pm \frac{H}{2}$	$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r} \frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0$	$\phi = A \cos \frac{\pi z}{H} J_0 \left(\frac{2.405}{R} r \right)$ $B^2 = \frac{\pi^2}{H^2} + \frac{(2.405)^2}{R^2}$	$\frac{\pi^2}{H^2} + \frac{(2.405)^2}{R^2} = B^2$ (For minimum critical volume $R = \frac{2.945}{B}$, $H = 1.847R = \frac{5.441}{B}$, $V_{\min} = \frac{148}{B^3}$)
Rectangular parallelepiped of sides = a , b , and c ; origin of coordinates at center	Planes $x = \pm \frac{a}{2}$ $y = \pm \frac{b}{2}$ $z = \pm \frac{c}{2}$	$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0$	$\phi = A \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \cos \frac{\pi z}{c}$ $B^2 = \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2}$	$\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2} = B^2$ (For minimum critical volume $a = b = c = \frac{\sqrt{3}\pi}{B} = \frac{5.441}{B}$, $V_{\min} = \frac{161}{B^3}$)
Sphere of radius R	$r = R$	$\frac{d^2\phi}{dr^2} + \frac{2}{r} \frac{d\phi}{dr} + B^2\phi = 0$	$\phi = \frac{A}{r} \sin \frac{\pi r}{R}$ $B^2 = \frac{\pi^2}{R^2}$	$R = \frac{\pi}{B}$ $V_{\text{crit}} = \frac{4}{3} \pi R^3 = \frac{4\pi^4}{3B^3} = \frac{130}{B^3}$

When the condition is added that the flux shall not be negative anywhere in the slab, all conditions are satisfied only by a slab of extrapolated half thickness $a/2 = \pi/2B$.

Thus the multiplying slab is critical if its extrapolated thickness a is just equal to π/B , where B is the solution of the characteristic equation (76).† The neutron flux will be constant in time, and its spatial distribution will be given by

$$\phi = \phi_0 \cos Bx = \phi_0 \cos \frac{\pi}{a} x$$

where ϕ_0 is the arbitrary value of the flux at $x = 0$, the center plane of the slab.

6.2 Rectangular Parallelepiped

The expanded form of the wave equation is

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} + B^2\phi = 0$$

† The physically meaningful expressions for $\bar{P}_\infty(E, B^2)$ have such form that the characteristic equation has a single solution for B^2 which is real and positive. Negative roots may, however, be present. In slab geometry they lead to solutions of the wave equation involving sinh and cosh (or exponential) terms. In the case of the single-region reactor these solutions are eliminated by the boundary conditions, either because they cannot be made to go to zero at the reactor boundary or because they give discontinuities in neutron current at the center of the reactor. These arguments do not hold, however, in the case of a multiregion reactor (see, for example, Art. 7).

If the origin of coordinates is taken at the center of the parallelepiped, the symmetric solution is

$$\phi = \phi_0 \cos \kappa_x x \cos \kappa_y y \cos \kappa_z z$$

where (cf. Art. 2.26)

$$\kappa_x^2 + \kappa_y^2 + \kappa_z^2 = B^2$$

Note that the solution is just a product of three slab solutions. This possibility of separating the solution into independent geometrical components is characteristic of bare reactors.

If the extrapolated dimensions of the parallelepiped are a , b , c in the x , y , and z directions, respectively,

$$\kappa_x^2 = \frac{\pi^2}{a^2} \quad \kappa_y^2 = \frac{\pi^2}{b^2} \quad \kappa_z^2 = \frac{\pi^2}{c^2}$$

and the reactor is critical if the dimensions are such that

$$\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} + \frac{\pi^2}{c^2} = B^2$$

Thus, the reactor can be made critical by proper adjustment of any one of the dimensions, the other two being fixed arbitrarily provided they exceed a certain minimum value (the value for which the third must become infinite).

6.3 Other Shapes

Table 5 gives the expanded form of the wave equation, its solution, and the critical dimensions for bare homogeneous reactors of various shapes.

6.4 The Absolute Value of the Thermal-neutron Flux

This value in a thermal reactor is given by

$$\bar{\phi} = 1.035 \times 10^{13} \frac{A}{\bar{\sigma}_f E} \frac{P}{M_f} \quad (96)$$

where $\bar{\phi}$ = average thermal flux over the reactor core volume

A = atomic weight of fissionable material in reactor

$\bar{\sigma}_f$ = microscopic fission cross section of fissionable material, averaged over thermal energy spectrum, barns

E = average energy liberation per fission, Mev

P/M_f = ratio of operating power of reactor to fissionable material content of the reactor, watts/g or kw/kg

If the fissionable isotope is U^{235} , and if $\bar{\sigma}_f$ is taken, roughly, as 500 barns and E as 200 Mev, †

$$\bar{\phi} = 2.4 \times 10^{10} \frac{P}{M_f} \quad (97)$$

For an effectively homogeneous reactor the above expressions give the average flux over the core volume. For a lumped reactor they give the average thermal flux in the fuel.

6.5 The Absolute Value of the Fast-neutron Flux

This value in a reactor is not often a very useful concept, since the total fast flux includes neutrons of such widely varying energies. For rough estimates the value of the two-group fast-neutron flux may be used. In a bare reactor (or far from the

† This value is only an approximate one. If an accurate value is required, it must be evaluated for the specific reactor in question, taking into account the leakage of neutrinos and some γ and neutron energy from the reactor, as well as the production of extra energy by the nonfission absorption of neutrons. Typical values lie between 190 and 200 Mev.

reflector in a large reflected reactor) the two-group fast flux is everywhere proportional to the thermal flux in the ratio

$$\frac{\phi_f}{\phi_s} = \frac{k \Sigma_{as}}{p \Sigma_{af}} \frac{1}{1 + L_f^2 B^2} \quad (98)$$

where Σ_s is the macroscopic thermal absorption cross section, L_f^2 is the slowing-down area, B^2 is the geometric buckling, and Σ_{af} is the fictitious slowing-down cross section ($= D_f/L_f^2$).

Although this equation is convenient to use, since the absolute value of ϕ_s is usually known, a more straightforward expression for ϕ_f is

$$\phi_f = \frac{\nu \times \text{No. fissions}/(\text{cm}^3)(\text{sec})}{\Sigma_{af}(1 + L_f^2 B^2)}$$

Thus, although the thermal flux depends on the specific power (power per unit mass of fissionable isotope), the fast flux depends only upon the number of fission neutrons produced per unit volume per second, the slowing-down cross section, and the fast leakage $1/(1 + L_f^2 B^2)$.

6.6 The Fractional Leakage (\mathcal{L}) of Neutrons

This value from a critical reactor is given by

$$\mathcal{L} = \frac{\text{No. neutrons leaking/sec}}{\text{No. fission neutrons born/sec}} = \frac{k - 1}{k} \quad (99)$$

For a bare thermal reactor, if it is assumed that the leakage of neutrons in the energy interval between resonance absorption and thermal can be neglected, the leakage can be divided into fast and thermal components (see Art. 5):

$$\begin{aligned} \frac{\text{No. fast neutrons leaking/sec}}{\text{No. fission neutrons born/sec}} &= 1 - \frac{\bar{P}_\infty}{p}(E_s, B^2) \\ &= 1 - e^{-\tau B^2} \quad \text{for Fermi slowing down} \\ &= \frac{L_f^2 B^2}{1 + L_f^2 B^2} \quad \text{for "two-group" slowing down} \end{aligned} \quad (100)$$

$$\frac{\text{No. thermal neutrons leaking/sec}}{\text{No. fission neutrons born/sec}} = \frac{L^2 B^2}{1 + L^2 B^2} \bar{P}_\infty(E_s, B^2) \quad (101)$$

A further useful leakage relation is

$$\frac{\text{No. thermal neutrons leaking/sec}}{\text{No. thermal neutrons absorbed/sec}} = L^2 B^2 \quad (102)$$

7 REFLECTED REACTORS

A reflector is ordinarily a mass of material having $k < 1$, which is placed around the outside of the multiplying core of the reactor to reduce the leakage of neutrons out of the core. This reduction of leakage has the two advantageous effects of reducing the quantity of fissionable material required for criticality and of causing more nearly uniform flux and power distributions in the reactor core. The most effective reflector material will be that which has the largest albedo for neutrons of the energy-distribution characteristic of the reactor in question. The reflector is often made of moderating material. In this case it returns neutrons to the reactor core at considerably reduced energies and modifies the energy spectrum near the core-reflector boundary.

Except in the special case for which the diffusion properties of the core and reflector are identical for all epithermal energies, reflected reactors can be solved only by one

Table 6. Definitions and Symbols Used in Arts. 7 and 8

Dimensions:

T, R_1, R_2, a, b = Dimensions of reactor regions. See Figs. 4 to 9.

Variables:

ρ = Dimensional variable of flux distributions; measured from inner dimension of each region in a slab, measured from center of core for cylinder and sphere
 ζ = Vector whose four components are, in order, fast flux, thermal flux, negative fast current, and negative thermal current at an internal boundary

M, N, V, W, X = Functional values of flux distributions

A, B, C, E = Numerical coefficients

Characteristic constants of the reactor:

k = Multiplication constant in the core material

u^2 = For any region, the contribution of the leakage from unreflected boundaries of the region to the total buckling

= $(\pi/a)^2$ for an unreflected slab of thickness a

= $(2.405/R)^2$ for an unreflected cylinder of radius R

κ_f^2 = Reciprocal of slowing-down area = $1/L_f^2$

κ_s^2 = Reciprocal of diffusion area = $1/L_s^2$

$\mu_f = \sqrt{\kappa_f^2 + u^2}$

$\mu_s = \sqrt{\kappa_s^2 + u^2}$

Σ_s = Macroscopic thermal absorption cross section

Σ_f = Macroscopic slowing-down cross section = D_f/L_f^2

D_s = Diffusion coefficient for thermal neutrons

D_f = Diffusion coefficient for fast neutrons

B^2 = Material buckling of any region

B_1^2 = "Fundamental" buckling in the core,

$$-\frac{1}{2}(\kappa_f^2 + \kappa_s^2) + \frac{1}{2}\sqrt{(\kappa_f^2 + \kappa_s^2)^2 + 4\kappa_f^2\kappa_s^2(k-1)}$$

$B_1'^2$ = "Transient" buckling in the core,

$$-\frac{1}{2}(\kappa_f^2 + \kappa_s^2) - \frac{1}{2}\sqrt{(\kappa_f^2 + \kappa_s^2)^2 + 4\kappa_f^2\kappa_s^2(k-1)}$$

$l = \sqrt{B_1^2 - u^2}$

$m = \sqrt{-B_1'^2 + u^2}$

S = Coupling coefficient between fast and thermal fluxes = $(\Sigma_s + D_s B^2)/\Sigma_f$ in general

$S = (\Sigma_s + D_s B_1^2)/\Sigma_f$ } in the core

$S' = (\Sigma_s + D_s B_1'^2)/\Sigma_f$ } in the core

$S = 0$ } in the reflector

$S' = (\Sigma_s + D_s \kappa_f^2)/\Sigma_f$ } in the reflector

Matrices:

Q = 4×2 matrix constructed for the outer reflector

Y_n = 4×4 matrix constructed for the n th intermediate region

Y_{core} = 2×4 matrix constructed for the core

Subscripts:

f = Denotes fast group of neutrons

s = Denotes thermal group of neutrons

of the group methods. This is because other descriptions of the slowing-down distribution provide no means for matching the neutron fluxes and currents at the core-reflector boundary over the epithermal energy distribution. In this and the following sections the two-group formulation will be used. The treatment for larger numbers of groups is similar but usually becomes too cumbersome for hand computation.

The mathematical treatment is similar for the various geometrical cases of reactors reflected in one dimension. One of these will be presented in some detail (Art. 7.1). In the other cases only the final equations will be given (Arts. 7.2 through 7.4).

7.1 Cylinder of Finite Length with Radial Reflector (Fig. 5)

This case will be worked out in some detail to illustrate the method. The reflector has $k = 0$.

The diffusion equations for the fast and thermal-neutron fluxes have been given previously [Eqs. (88) and (89)]. To minimize the complexity of the notation in the following treatment, some of the subscripts will be omitted and Eqs. (88) and (89) will be written as

$$D_f \nabla^2 \phi_f - \Sigma_f \phi_f + k \Sigma_s \phi_s = 0 \quad (103)$$

$$D_s \nabla^2 \phi_s - \Sigma_s \phi_s - \Sigma_f \phi_f = 0 \quad (104)$$

It is to be understood that Σ_s in these equations designates the thermal absorption cross section and Σ_f the fictitious fast absorption cross section (see Table 6 for a complete list of the notation used in Secs. 7 and 8). It has been assumed in Eqs. (103) and (104) that resonance absorption is negligible. If this is not the case p can be included as indicated in Eqs. (88) and (89).

In the core, $k > 1$ for a critical reactor; in the reflector, $k = 0$. In either case, the solutions are the solutions of the wave equations

$$\nabla^2 \phi_f + B^2 \phi_f = 0 \quad (105)$$

$$\nabla^2 \phi_s + B^2 \phi_s = 0 \quad (106)$$

the solutions being coupled by the condition that the relation between fast and thermal flux, as defined by Eq. (104) [or, alternatively, by Eq. (103)], must be preserved:

$$S = \frac{\phi_f}{\phi_s} = \frac{\Sigma_s + D_s B^2}{\Sigma_f} \quad (107)$$

and the solutions being further restricted by the requirement that the characteristic equation be satisfied:

$$\left(1 + \frac{B^2}{\kappa_s^2}\right) \left(1 + \frac{B^2}{\kappa_f^2}\right) - k = 0 \quad (108)$$

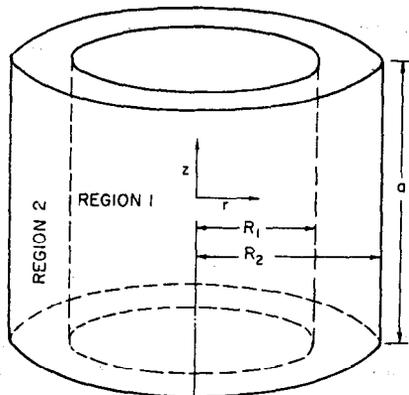


FIG. 5.

This characteristic equation is a quadratic in B^2 , which defines two values of B^2 (B^2 and B'^2) in each region. In the core, which will be characterized by the subscript 1,

$$B_1^2 = \frac{-(\kappa_{1f}^2 + \kappa_{1s}^2) + \sqrt{(\kappa_{1f}^2 + \kappa_{1s}^2)^2 + 4\kappa_{1f}^2 \kappa_{1s}^2 (k - 1)}}{2} \quad (109)$$

$$B_1'^2 = \frac{-(\kappa_{1f}^2 + \kappa_{1s}^2) - \sqrt{(\kappa_{1f}^2 + \kappa_{1s}^2)^2 + 4\kappa_{1f}^2 \kappa_{1s}^2 (k - 1)}}{2} \quad (110)$$

In the reflector, which will be characterized by the subscript 2,

$$B_2^2 = -\kappa_{2s}^2 \quad (111)$$

$$B_2'^2 = -\kappa_{2f}^2 \quad (112)$$

If it is assumed that the wave equation is separable in both core and reflector, so that the solutions are the products of independent functions of z and r , then the solution for each flux (fast and thermal) in each region (core and reflector) is of the form

$$\phi = \cos \frac{\pi z}{a} F(r) \quad (113)$$

where $F(r)$ is the solution of the equation

$$\frac{d^2 F}{dr^2} + \frac{1}{r} \frac{dF}{dr} + \left[B^2 - \left(\frac{\pi}{a} \right)^2 \right] F = 0 \quad (114)$$

Since there are two values of B^2 in both core and reflector [Eqs. (109) to (112)], a total of four equations of this type must be satisfied to define each F completely in both regions, thus:

$$\left. \begin{aligned} \frac{d^2 F_{1f}}{dr^2} + \frac{1}{r} \frac{dF_{1f}}{dr} + l^2 F_{1f} &= 0 \\ \frac{d^2 F_{1s}}{dr^2} + \frac{1}{r} \frac{dF_{1s}}{dr} - m^2 F_{1s} &= 0 \end{aligned} \right\} \text{core} \quad (115)$$

$$\left. \begin{aligned} \frac{d^2 F_{2f}}{dr^2} + \frac{1}{r} \frac{dF_{2f}}{dr} - n^2 F_{2f} &= 0 \\ \frac{d^2 F_{2s}}{dr^2} + \frac{1}{r} \frac{dF_{2s}}{dr} - p^2 F_{2s} &= 0 \end{aligned} \right\} \text{reflector} \quad (116)$$

$$\left. \begin{aligned} \frac{d^2 F_{2f}}{dr^2} + \frac{1}{r} \frac{dF_{2f}}{dr} - \mu_f^2 F_{2f} &= 0 \\ \frac{d^2 F_{2s}}{dr^2} + \frac{1}{r} \frac{dF_{2s}}{dr} - \mu_s^2 F_{2s} &= 0 \end{aligned} \right\} \text{reflector} \quad (117)$$

$$\left. \begin{aligned} \frac{d^2 F_{2f}}{dr^2} + \frac{1}{r} \frac{dF_{2f}}{dr} - \mu_f^2 F_{2f} &= 0 \\ \frac{d^2 F_{2s}}{dr^2} + \frac{1}{r} \frac{dF_{2s}}{dr} - \mu_s^2 F_{2s} &= 0 \end{aligned} \right\} \text{reflector} \quad (118)$$

with identical equations for F_{1s} and F_{2s} . In the above equations the signs of the constants l^2 , m^2 , μ_f^2 , and μ_s^2 are so chosen that the constants themselves are inherently positive:

$$l^2 = B_1^2 - \left(\frac{\pi}{H}\right)^2 \quad (119)$$

$$-m^2 = B_1'^2 - \left(\frac{\pi}{H}\right)^2 \quad (120)$$

$$-\mu_f^2 = -\kappa_{2f}^2 - \left(\frac{\pi}{H}\right)^2 \quad (121)$$

$$-\mu_s^2 = -\kappa_{2s}^2 - \left(\frac{\pi}{H}\right)^2 \quad (122)$$

In the core, the complete solutions for the fast and thermal fluxes are the sums of those solutions of Eqs. (115) and (116) which are finite for $r = 0$:

$$F_{1s} = AJ_0(lr) + CI_0(mr) \quad (123)$$

$$F_{1f} = AS_1 J_0(lr) + CS_1' I_0(mr) \quad (124)$$

In the reflector the solutions for fast and thermal fluxes are the sums of the solutions of Eqs. (117) and (118).

$$F_{2s} = EI_0(\mu_s r) + GK_0(\mu_s r) + HI_0(\mu_f r) + MK_0(\mu_f r) \quad (125)$$

$$F_{2f} = S_2 EI_0(\mu_s r) + S_2 GK_0(\mu_s r) + S_2' HI_0(\mu_f r) + S_2' MK_0(\mu_f r) = S_2' HI_0(\mu_f r) + S_2' MK_0(\mu_f r) \quad (126)$$

In these equations, A , C , E , G , H , and M are constants to be determined by the boundary conditions and S_1 , S_1' , S_2 , and S_2' are the coupling coefficients [Eq. (107)] appropriate to the region under consideration and the value of B^2 :

$$S_1 = \frac{\Sigma_{1s} + D_{1s} B_1^2}{\Sigma_{1f}} \quad (127)$$

$$S_1' = \frac{\Sigma_{1s} + D_{1s} B_1'^2}{\Sigma_{1f}} \quad (128)$$

$$S_2 = \frac{\Sigma_{2s} - D_{2s} \kappa_{2s}^2}{\Sigma_{2f}} = 0 \quad (129)$$

$$S_2' = \frac{\Sigma_{2s} - D_{2s} \kappa_{2f}^2}{\Sigma_{2f}} \quad (130)$$

Equations (123) to (126) contain six undetermined coefficients (A , C , E , G , H , and M). Five of these, plus the criticality condition, are determined by the following six boundary conditions:

$$F_{2f}(R_2) = 0 \quad (131)$$

$$F_{2s}(R_2) = 0 \quad (132)$$

$$F_{1f}(R_1) = F_{2f}(R_1) \quad (133)$$

$$F_{1s}(R_1) = F_{2s}(R_1) \quad (134)$$

$$D_{1f} \frac{dF_{1f}}{dr}(R_1) = D_{2f} \frac{dF_{2f}}{dr}(R_1) \quad (135)$$

$$D_{1s} \frac{dF_{1s}}{dr}(R_1) = D_{2s} \frac{dF_{2s}}{dr}(R_1) \quad (136)$$

Applying conditions (131) and (132) to Eqs. (125) and (126) there result

$$\begin{aligned} EI_0(\mu_s R_2) + GK_0(\mu_s R_2) + HI_0(\mu_f R_2) + MK_0(\mu_f R_2) &= 0 \\ S_2' HI_0(\mu_f R_2) + S_2' MK_0(\mu_f R_2) &= 0 \\ E &= -G \frac{K_0(\mu_s R_2)}{I_0(\mu_s R_2)} \end{aligned} \quad (137)$$

$$H = -M \frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} \quad (138)$$

Using these relations, and applying conditions (133) to (136) to Eqs. (123) to (126), the four simultaneous equations result:

$$\begin{aligned} AJ_0(lR_1) + CI_0(mR_1) + G \left[\frac{K_0(\mu_s R_2)}{I_0(\mu_s R_2)} I_0(\mu_s R_1) - K_0(\mu_s R_1) \right] \\ + M \left[\frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_0(\mu_f R_1) - K_0(\mu_f R_1) \right] &= 0 \end{aligned} \quad (139)$$

$$AS_1 J_0(lR_1) + CS_1' I_0(mR_1) + S_2' M \left[\frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_0(\mu_f R_1) - K_0(\mu_f R_1) \right] = 0 \quad (140)$$

$$\begin{aligned} D_{1s} A I J_1(lR_1) - D_{1s} C m I_1(mR_1) - D_{2s} \mu_s G \left[\frac{K_0(\mu_s R_2)}{I_0(\mu_s R_2)} I_1(\mu_s R_1) + K_1(\mu_s R_1) \right] \\ - D_{2s} \mu_f M \left[\frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_1(\mu_f R_1) + K_1(\mu_f R_1) \right] &= 0 \end{aligned} \quad (141)$$

$$\begin{aligned} D_{1f} A S_1 J_1(lR_1) - D_{1f} C S_1' m I_1(mR_1) \\ - S_2' D_{2f} \mu_f M \left[\frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_1(\mu_f R_1) + K_1(\mu_f R_1) \right] &= 0 \end{aligned} \quad (142)$$

These four linear homogeneous equations, of the form

$$\begin{aligned} k_{11}A + k_{12}C + k_{13}G + k_{14}M &= 0 \\ k_{21}A + k_{22}C + k_{23}G + k_{24}M &= 0 \\ k_{31}A + k_{32}C + k_{33}G + k_{34}M &= 0 \\ k_{41}A + k_{42}C + k_{43}G + k_{44}M &= 0 \end{aligned} \quad (143)$$

will have a nontrivial solution for A , C , G , and M provided the determinant

$$D = \begin{vmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{vmatrix} = 0 \quad (144)$$

Physically, this condition is the condition of *criticality* for the reactor. In general, for an arbitrary choice of reactor characteristics, the determinant will have some nonzero value. A practical method for finding a condition of criticality is to compute the value of the determinant D for a number of different values of one of the reactor characteristics (usually R_1 , H , k , or uranium density), to plot the value of D as a function of the value of the variable characteristic, and thus graphically to determine the value of the characteristic for which $D = 0$.

Once the condition of criticality has been established, the set of equations (143) defines the relationships among the constants A , C , G , and M . If one of the constants (most conveniently, A) is set arbitrarily, the remaining three may be determined in terms of it by three of the equations (143), and the constants E and H can be specified in the same terms by Eqs. (137) and (138). Substitution of these values into Eqs. (123) to (126) specifies the thermal and epithermal flux in the core and reflector, except for an arbitrary multiplier. This multiplier, which specifies the absolute level of the flux, can be determined if the power at which the reactor is operating is known (see Art. 6.4).

7.2 Finite Cylinder Reflected on the Ends (Fig. 6)

The fluxes are given by

$$\phi_s = J_0 \frac{2.405r}{R} F_s(z) \tag{145}$$

$$\phi_f = J_0 \frac{2.405r}{R} F_f(z) \tag{146}$$

where F_s and F_f are given by equations of the form

$$\frac{d^2F}{dz^2} + \left[B^2 - \left(\frac{2.405}{R} \right)^2 \right] F = 0 \tag{147}$$

which have the solutions

$$\begin{aligned} F_{1s} &= A \cos lz + C \cosh mz \\ F_{1f} &= S_1 A \cos lz + S_1' C \cosh mz \\ F_{2s} &= E \cosh \mu_s \zeta + G \cosh \mu_f \zeta \\ F_{2f} &= S_2' G \cosh \mu_f \zeta \end{aligned} \tag{148}$$

where

$$\begin{aligned} l^2 &= B_1^2 - \left(\frac{2.405}{R} \right)^2 & -m^2 &= B_1'^2 - \left(\frac{2.405}{R} \right)^2 \\ -\mu_f^2 &= -\kappa_{2f}^2 - \left(\frac{2.405}{R} \right)^2 & -\mu_s^2 &= -\kappa_{2s}^2 - \left(\frac{2.405}{R} \right)^2 \end{aligned} \tag{149}$$

The coefficients $A, C, E,$ and G are related by the equations

$$\begin{aligned} A \cos \frac{lH}{2} + C \cosh \frac{mH}{2} - E \cosh \mu_s T - G \cosh \mu_f T &= 0 \\ S_1 A \cos \frac{lH}{2} + S_1' C \cosh \frac{mH}{2} - S_2' G \cosh \mu_f T &= 0 \\ -lD_{1s} A \sin \frac{lH}{2} + mD_{1s} C \sinh \frac{mH}{2} + \mu_s D_{2s} E \sinh \mu_s T + \mu_f D_{2s} G \sinh \mu_f T &= 0 \\ -lD_{1f} S_1 A \sin \frac{lH}{2} + mD_{1f} S_1' C \sinh \frac{mH}{2} + \mu_f D_{2f} S_2' G \sinh \mu_f T &= 0 \end{aligned} \tag{150}$$

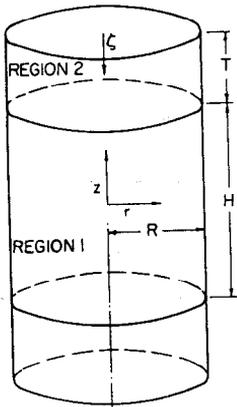


FIG. 6.

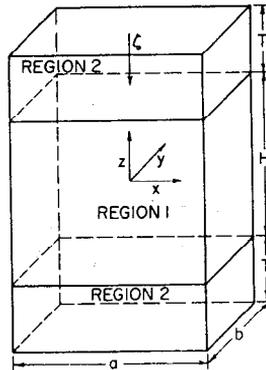


FIG. 7.

The methods used for Eqs. (139) through (142) can be applied to these equations to find the condition of criticality and to determine three of the four coefficients $A, C, E,$ and $G.$

7.3 Rectangular Parallelepiped Reflected on Two Opposite Faces (Fig. 7)

The fluxes are given by

$$\phi_s = \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} F_s(z) \quad (151)$$

$$\phi_f = \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} F_f(z) \quad (152)$$

where F_s and F_f are given by equations of the form

$$\frac{d^2 F}{dz^2} + \left[B^2 - \left(\frac{\pi}{a} \right)^2 - \left(\frac{\pi}{b} \right)^2 \right] F = 0 \quad (153)$$

The solutions for F are the same as those given in Eqs. (148) above, and the equations which determine the coefficients and the criticality condition are Eqs. (150) above, with the following changes in the definition of symbols:

$$\begin{aligned} l^2 &= B_1^2 - \left(\frac{\pi}{a} \right)^2 - \left(\frac{\pi}{b} \right)^2 & -m^2 &= B_1'^2 - \left(\frac{\pi}{a} \right)^2 - \left(\frac{\pi}{b} \right)^2 \\ -\mu_f^2 &= -\kappa_f^2 - \left(\frac{\pi}{a} \right)^2 - \left(\frac{\pi}{b} \right)^2 & -\mu_s^2 &= -\kappa_s^2 - \left(\frac{\pi}{a} \right)^2 - \left(\frac{\pi}{b} \right)^2 \end{aligned} \quad (154)$$

7.4 Reflected Sphere (Fig. 8)

The fluxes are given by

$$\begin{aligned} \phi_{1s} &= \frac{1}{r} (A \sin B_1 r + C \sinh B_1' r) \\ \phi_{1f} &= \frac{1}{r} (S_1 A \sin B_1 r + S_1' C \sinh B_1' r) \\ \phi_{2s} &= \frac{1}{r} [E (\cosh \kappa_s r - \coth \kappa_s R_2 \sinh \kappa_s r) + H (\cosh \kappa_f r - \coth \kappa_f R_2 \sinh \kappa_f r)] \\ \phi_{2f} &= \frac{S_2'}{r} H (\cosh \kappa_f r - \coth \kappa_f R_2 \sinh \kappa_f r) \end{aligned} \quad (155)$$

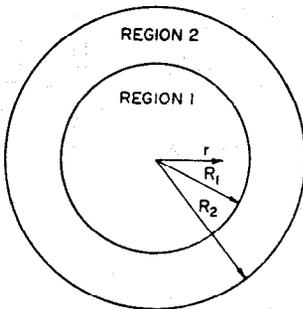


FIG. 8.

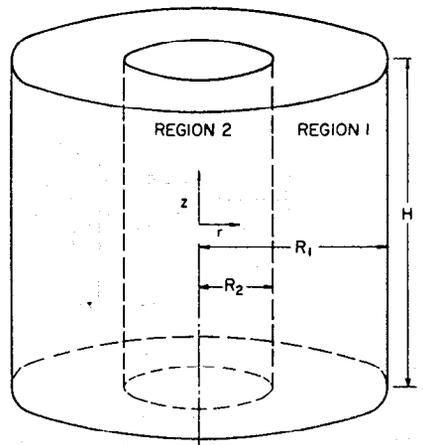


FIG. 9.

and the coefficients A , C , E , H are related by the equations

$$\begin{aligned}
 & A \sin B_1 R_1 + C \sinh B_1' R_1 + E(\coth \kappa_s R_2 \sinh \kappa_s R_1 - \cosh \kappa_s R_1) \\
 & \quad + H(\coth \kappa_f R_2 \sinh \kappa_f R_1 - \cosh \kappa_f R_1) = 0 \\
 & AS_1 \sin B_1 R_1 + CS_1' \sinh B_1' R_1 + HS_2'(\coth \kappa_f R_2 \sinh \kappa_f R_1 - \cosh \kappa_f R_1) = 0 \\
 & AD_{1s} \left(B_1 \cos B_1 R_1 - \frac{1}{R_1} \sin B_1 R_1 \right) + CD_{1s} \left(B_1' \cosh B_1' R_1 - \frac{1}{R_1} \sinh B_1' R_1 \right) \\
 & \quad + ED_{2s} \left(\kappa_s \coth \kappa_s R_2 \cosh \kappa_s R_1 - \kappa_s \sinh \kappa_s R_1 \right. \\
 & \quad \left. - \frac{1}{R_1} \coth \kappa_s R_2 \sinh \kappa_s R_1 + \frac{1}{R_1} \cosh \kappa_s R_1 \right) \\
 & \quad + HD_{2s} \left(\kappa_f \coth \kappa_f R_2 \cosh \kappa_f R_1 - \kappa_f \sinh \kappa_f R_1 \right. \\
 & \quad \left. - \frac{1}{R_1} \coth \kappa_f R_2 \sinh \kappa_f R_1 + \frac{1}{R_1} \cosh \kappa_f R_1 \right) = 0 \tag{156} \\
 & AD_{1f} \left(S_1 B_1 \cos B_1 R_1 - \frac{1}{R_1} S_1 \sin B_1 R_1 \right) \\
 & \quad + CD_{1f} \left(S_1' B_1' \cosh B_1' R_1 - \frac{1}{R_1} S_1' \sinh B_1' R_1 \right) \\
 & \quad + HD_{2f} \left(S_2' \kappa_f \coth \kappa_f R_2 \cosh \kappa_f R_1 - S_2' \kappa_f \sinh \kappa_f R_1 \right. \\
 & \quad \left. - \frac{S_2'}{R_1} \coth \kappa_f R_2 \sinh \kappa_f R_1 + \frac{S_2'}{R_1} \cosh \kappa_f R_1 \right) = 0
 \end{aligned}$$

and again, these equations may be used to determine the criticality condition and to fix three of the coefficients, as in the case of Eqs. (139) through (142).

7.5 Bare Cylindrical Reactor Containing Central "Reflector" or Absorber (Fig. 9)

This case occurs in computing the effects of centrally located control elements or sometimes in computing the effects of experiments inserted in the reactor. The quantity k_2 is assumed to be < 1 , but for the sake of generality is not taken to be necessarily zero.

The fluxes are given by

$$\begin{aligned}
 \phi_s &= \cos \frac{\pi z}{H} F_s(r) \\
 \phi_f &= \cos \frac{\pi z}{H} F_f(r)
 \end{aligned} \tag{157}$$

where F_s and F_f are given by equations of the form

$$\frac{d^2 F}{dr^2} + \frac{1}{r} \frac{dF}{dr} + \left[B^2 - \left(\frac{\pi}{H} \right)^2 \right] F = 0 \tag{158}$$

and B^2 has two values for each geometrical region:

$$\begin{aligned}
 B_1^2 &= \frac{-(\kappa_{1f}^2 + \kappa_{1s}^2) + \sqrt{(\kappa_{1f}^2 + \kappa_{1s}^2)^2 + 4\kappa_{1f}^2 \kappa_{1s}^2 (k_1 - 1)}}{2} \\
 B_1'^2 &= \frac{-(\kappa_{1f}^2 + \kappa_{1s}^2) - \sqrt{(\kappa_{1f}^2 + \kappa_{1s}^2)^2 + 4\kappa_{1f}^2 \kappa_{1s}^2 (k_1 - 1)}}{2}
 \end{aligned} \tag{159}$$

$$B_2^2 = \frac{-(\kappa_{2f}^2 + \kappa_{2s}^2) + \sqrt{(\kappa_{2f}^2 + \kappa_{2s}^2)^2 + 4\kappa_{2f}^2\kappa_{2s}^2(k_2 - 1)}}{2}$$

$$B_2'^2 = \frac{-(\kappa_{2f}^2 + \kappa_{2s}^2) - \sqrt{(\kappa_{2f}^2 + \kappa_{2s}^2)^2 + 4\kappa_{2f}^2\kappa_{2s}^2(k_2 - 1)}}{2} \quad (160)$$

The quantities l^2 , m^2 , μ_f^2 , and μ_s^2 are defined as

$$l^2 = B_1^2 - \left(\frac{\pi}{H}\right)^2 \quad -m^2 = B_1'^2 - \left(\frac{\pi}{H}\right)^2$$

$$-\mu_s^2 = B_2^2 - \left(\frac{\pi}{H}\right)^2 \quad -\mu_f^2 = B_2'^2 - \left(\frac{\pi}{H}\right)^2 \quad (161)$$

The solutions of the four equations of the form (158) are, then,

$$F_{1s} = AJ_0(lr) + CY_0(lr) + EI_0(mr) + GK_0(mr)$$

$$F_{1f} = S_1[AJ_0(lr) + CY_0(lr)] + S_1'[EI_0(mr) + GK_0(mr)]$$

$$F_{2s} = HI_0(\mu_s r) + MI_0(\mu_f r)$$

$$F_{2f} = S_2HI_0(\mu_s r) + S_2'MI_0(\mu_f r) \quad (162)$$

where

$$S_1 = \frac{\Sigma_{1s} + D_{1s}B_1^2}{\Sigma_{1f}} \quad S_1' = \frac{\Sigma_{1s} + D_{1s}B_1'^2}{\Sigma_{1f}}$$

$$S_2 = \frac{\Sigma_{2s} - D_{2s}B_2^2}{\Sigma_{2f}} \quad S_2' = \frac{\Sigma_{2s} - D_{2s}B_2'^2}{\Sigma_{2f}} \quad (163)$$

Applying the boundary conditions

$$F_{1f}(R_1) = 0 \quad F_{1s}(R_1) = 0$$

$$F_{1f}(R_2) = F_{2f}(R_2) \quad F_{1s}(R_2) = F_{2s}(R_2) \quad (164)$$

$$D_{1f} \frac{dF_{1f}}{dr}(R_2) = D_{2f} \frac{dF_{2f}}{dr}(R_2) \quad D_{1s} \frac{dF_{1s}}{dr}(R_2) = D_{2s} \frac{dF_{2s}}{dr}(R_2)$$

the constants A and E can be evaluated in terms of C and G , respectively:

$$A = -C \frac{Y_0(lR_1)}{J_0(lR_1)} \quad E = -G \frac{K_0(mR_1)}{I_0(mR_1)} \quad (165)$$

and the remaining constants C , G , H , and M are related by the equations

$$C \left[-\frac{Y_0(lR_1)}{J_0(lR_1)} J_0(lR_2) + Y_0(lR_2) \right] + G \left[K_0(mR_2) - \frac{K_0(mR_1)}{I_0(mR_1)} I_0(mR_2) \right]$$

$$- HI_0(\mu_s R_2) - MI_0(\mu_f R_2) = 0$$

$$S_1 C \left[-\frac{Y_0(lR_1)}{J_0(lR_1)} J_0(lR_2) + Y_0(lR_2) \right] + S_1' G \left[K_0(mR_2) - \frac{K_0(mR_1)}{I_0(mR_1)} I_0(mR_2) \right]$$

$$- S_2 HI_0(\mu_s R_2) - S_2' MI_0(\mu_f R_2) = 0$$

$$D_{1s} C \left[l \frac{Y_0(lR_1)}{J_0(lR_1)} J_1(lR_2) - l Y_1(lR_2) \right]$$

$$+ D_{1s} G \left[-m K_1(mR_2) - m \frac{K_0(mR_1)}{I_0(mR_1)} I_1(mR_2) \right]$$

$$- D_{2s} [H \mu_s I_1(\mu_s R_2) + M \mu_f I_1(\mu_f R_2)] = 0$$

$$S_1 D_{1f} C \left[l \frac{Y_0(lR_1)}{J_0(lR_1)} J_1(lR_2) - l Y_1(lR_2) \right]$$

$$+ S_1' D_{1f} G \left[-m K_1(mR_2) - m \frac{K_0(mR_1)}{I_0(mR_1)} I_1(mR_2) \right]$$

$$- D_{2f} [S_2 H \mu_s I_1(\mu_s R_2) + S_2' M \mu_f I_1(\mu_f R_2)] = 0 \quad (166)$$

The criticality condition can be determined and three of the constants can be evaluated from these four relations as explained in the case of Eqs. (139) through (142).

If the central region is a control rod, it is usually black to thermal neutrons, and the boundary condition on the thermal flux in region 1 is

$$\frac{1}{\phi_{s1}(R_2)} \frac{d\phi_{s1}}{dr}(R_2) = \frac{1}{\epsilon}$$

where ϵ is the extrapolation distance for thermal neutrons (see Art. 2.7). Often the rod material is a poor moderator, and a suitable boundary condition on the fast flux is

$$\frac{d\phi_{f1}}{dr}(R_2) = 0$$

When these conditions hold, the two constants C and G , which must be evaluated to describe the flux distribution in region 1, are determined by

$$\frac{1}{\epsilon} = \frac{Cl \left[\frac{Y_0(lR_1)}{J_0(lR_1)} J_1(lR_2) - Y_1(lR_2) \right] - Gm \left[\frac{K_0(mR_1)}{I_0(mR_1)} I_1(mR_2) + K_1(mR_2) \right]}{C \left[-\frac{Y_0(lR_1)}{J_0(lR_1)} J_0(lR_2) + Y_0(lR_2) \right] - G \left[\frac{K_0(mR_1)}{I_0(mR_1)} I_0(mR_2) - K_0(mR_2) \right]} \quad (167)$$

and

$$S_1 Cl \left[\frac{Y_0(lR_1)}{J_0(lR_1)} J_1(lR_2) - Y_1(lR_2) \right] - S_1 Gm \left[\frac{K_0(mR_1)}{I_0(mR_1)} I_1(mR_2) + K_1(mR_2) \right] = 0$$

7.6 The Reflector Saving

Given a reflected reactor known to be critical (either by calculation or experiment) whose core has dimensions a , b , and c in some coordinate system whose directions are specified by α , β , γ , we remove the reflector which extends in, say, the α direction and determine (by either calculation or experiment) how far the bare core would have to be extended in the α direction to achieve criticality. If this critical dimension of the bare core is a' , then we define the reflector saving in the α direction as $a' - a$. The dimension a' is called the equivalent bare dimension of the reactor.

If the reflector saving for a given core-reflector combination is known, the criticality calculations for the reactor can be made as though the reactor were bare, using the reflector saving in place of the usual extrapolation distance.

The value of the reflector saving is often relatively insensitive to changes in some of the important characteristics of the core. Furthermore, in many practical cases the reflector saving is a relatively small fraction of the equivalent bare dimension. These considerations make the reflector-saving concept a very useful one in practical reactor calculations. For example, if calculations are to be made over a wide range of reactor variables for design optimization, reflector savings can be calculated for only a few conditions and values for the remainder can be interpolated.

7.7 Reactors Reflected in More than One Dimension

There is no analytical solution for the reactor reflected in more than one dimension, although numerical methods (e.g., the relaxation methods) are possible. The usual method for practical calculations is to assume that the reflector saving in any given dimension is the same as that which would result if the reactor were bare and of the equivalent bare size in the other dimensions. Thus, for a three-dimensional problem one guesses first an equivalent bare value for each of two dimensions and solves the actual reflected case in the third dimension, thus determining the equivalent bare value for that dimension. The same process is applied in turn to each dimension, using the "correct" values for the equivalent bare dimensions as they are deter-

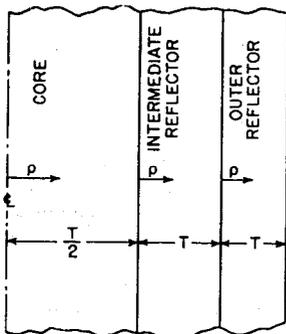
mined and guessed values when necessary. After a calculated result has been obtained for the equivalent bare value in each dimension, criticality can be determined by the bare reactor solutions. If, as the calculation proceeds, it is evident that some of the guesses made were very poor, it may be advisable to repeat the calculations using the improved values for the equivalent bare dimensions. Experience has shown that, as would be expected, this method gives accurate results for reasonably large reactors having reflector savings which are not large fractions of the equivalent bare dimensions.

An alternate method can be used if the core is of a compact shape and is identically reflected on all sides. The core and reflector are replaced by spherical regions of the same materials and of roughly the same geometrical buckling. The reflector saving is computed for this spherical assembly, and it is then assumed that the same reflector saving will apply to each dimension of the actual core-reflector assembly. This method involves less computation than the preceding and may give more accurate results if the reflector saving is a large fraction of the equivalent bare dimension.

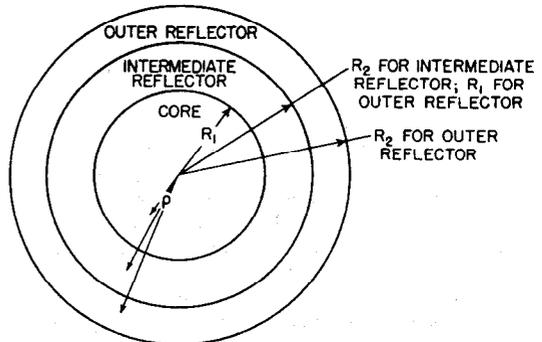
8 MATRIX SOLUTIONS FOR REFLECTED REACTORS

By Otto Schulze

The problem of the two-group reactor reflected in one dimension, which was treated in Art. 7, can also be solved by the application of matrix algebra. The results obtained can also be applied approximately to reactors reflected in more than one dimension by the same schemes outlined in Art. 7.



SLAB GEOMETRY



CYLINDRICAL OR SPHERICAL GEOMETRY

Fig. 10. Notation for multiple-reflector problems.

The matrix solutions are particularly useful for reflectors which consist of more than one region (Fig. 10), all of which are here assumed to be nonmultiplying. They were originally described by Garabedian and Householder (MonP-246) and were reduced to a straightforward computational procedure by Spinrad and Kurath.† The symbols used here are the same as those of Art. 7 (see Table 6).

8.1 General Procedure for Calculating Criticality

For each region there is a matrix whose elements are determined by the two-group constants and the dimensions of the region. For the outermost reflector the matrix is called the Q matrix. It has a simple form because the fluxes vanish at the outer (extrapolated) boundary. For each intermediate region n there is a matrix designated Y_n , while the matrix for the core region is symbolized by Y_{core} .

† This section is a condensation of Ref. 11. The report contains form sheets which are very useful if extensive computations are to be made.

After all the elements of all the matrices have been determined, one proceeds from the outer reflector through the intermediate regions in consecutive order to the core region, forming the product first of the outer reflector Q matrix multiplied into the Y_n matrix of the adjacent intermediate region. This product is then multiplied into the Y_n matrix of the next intermediate region, and so on, until finally a product is formed with the Y_{core} matrix. This final product is a 2×2 matrix. The critical condition is that the determinant of this final matrix be zero.

Criticality can be achieved either by varying a particular parameter (for instance, the dimension of the core) until the value of the determinant is reduced to zero or (as is more convenient if the nuclear constants, i.e., fuel loading, are to be varied) by setting the determinant equal to zero and solving for one of the functions $l \tan (lt)$, $l[J_0(lr)/J_1(lr)]$, or $l \cot (lr)$ (depending upon whether the geometry is a slab, cylinder, or sphere). In this notation l is the square root of the partial buckling. Knowing l , one can obtain the value of the total buckling B_1^2 . If B_1^2 is not the same as the test value, one iterates, using the new solution as a test value, repeating this process until convergence is obtained. The procedure is demonstrated in Art. 11.7.

8.2 The Q Matrix (Outer Reflector)

$$Q = \begin{vmatrix} 1 & 0 & \frac{1}{D_f} q(\mu_f) & 0 \\ 0 & 1 & q(\mu_f, \mu_s) & \frac{1}{D_s} q(\mu_s) \end{vmatrix}$$

where

$$q(\mu) = \begin{matrix} \text{Slab} & \text{Sphere} & \text{Cylinder} \\ \left| \frac{\tanh \mu T}{\mu} \right| & \left| \frac{R_1}{1 + \mu R_1 \coth \mu(R_2 - R_1)} \right| & \left| \frac{\frac{1}{\mu} \frac{K_0(\mu R_1)}{K_1(\mu R_1)}}{1 + \frac{K_0(\mu R_2) I_0(\mu R_1)}{K_1(\mu R_1) I_0(\mu R_2)}} \right| \end{matrix}$$

and

$$q(\mu_f, \mu_s) = \frac{1}{S'D_f} [q(\mu_f) - q(\mu_s)]$$

8.3 The Y_n Matrix (for Any Intermediate Region n)

$$Y_n = \begin{vmatrix} X_1(\mu_f) & 0 & \frac{1}{D_f} X_2(\mu_f) & 0 \\ X_1(\mu_f, \mu_s) & X_1(\mu_s) & X_2(\mu_f, \mu_s) & \frac{1}{D_s} X_2(\mu_s) \\ D_f X_3(\mu_f) & 0 & X_4(\mu_f) & 0 \\ X_3(\mu_f, \mu_s) & D_s X_3(\mu_s) & X_4(\mu_f, \mu_s) & X_4(\mu_s) \end{vmatrix}$$

where

$$\begin{matrix} \text{Slab} & \text{Sphere} & \text{Cylinder} \\ X_1(\mu) = \cosh \mu T & \left| \cosh \mu(R_2 - R_1) + \frac{1}{\mu R_1} \sinh \mu(R_2 - R_1) \right| & \left| \frac{\mu R_1 [I_0(\mu R_2) K_1(\mu R_1) + K_0(\mu R_2) I_1(\mu R_1)]}{R_1 [I_0(\mu R_2) K_0(\mu R_1) - K_0(\mu R_2) I_0(\mu R_1)]} \right| \\ X_2(\mu) = \frac{\sinh \mu T}{\mu} & \left| \frac{\sinh \mu(R_2 - R_1)}{\mu} \right| & \left| \frac{\mu^2 R_1 [I_1(\mu R_2) K_1(\mu R_1) - K_1(\mu R_2) I_1(\mu R_1)]}{-K_1(\mu R_2) I_0(\mu R_1)} \right| \\ X_3(\mu) = \mu \sinh \mu T & \left| \frac{1}{\mu R_1 R_2} [\mu(R_2 - R_1) \cosh \mu(R_2 - R_1) + (\mu^2 R_1 R_2 - 1) \sinh \mu(R_2 - R_1)] \right| & \left| \frac{\mu^2 R_1 [I_1(\mu R_2) K_1(\mu R_1) - K_1(\mu R_2) I_1(\mu R_1)]}{-K_1(\mu R_2) I_0(\mu R_1)} \right| \\ X_4(\mu) = \cosh \mu T & \left| \cosh \mu(R_2 - R_1) - \frac{1}{\mu R_2} \sinh \mu(R_2 - R_1) \right| & \left| \frac{\mu R_1 [I_1(\mu R_2) K_0(\mu R_1) + K_1(\mu R_2) I_0(\mu R_1)]}{+K_1(\mu R_2) I_0(\mu R_1)} \right| \end{matrix}$$

where A' is obtained from

$$QY_{\text{core}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with

$$A' = -\frac{a}{b}$$

or

$$A' = -\frac{c}{d} \quad (\text{check})$$

The outer edge values of the fluxes and currents are obtained from the matrix product; e.g.,

$$\|Y_{\text{core}}\| \begin{bmatrix} M(IR_1) \\ AN(mR_1) \end{bmatrix} = \|\xi\| = \begin{bmatrix} \phi_f \\ \phi_s \\ J_f \\ J_s \end{bmatrix}$$

Intermediate Region:

$$\phi_f(\rho) = AV(\mu_f\rho) + BW(\mu_f\rho)$$

$$\phi_s(\rho) = \frac{1}{S'}\phi_f(\rho) + CV(\mu_s\rho) + EW(\mu_s\rho)$$

	Slab	Sphere	Cylinder
$V(\mu\rho)$ $W(\mu\rho)$	$\left. \begin{matrix} \cosh \mu\rho \\ \sinh \mu\rho \end{matrix} \right\} \rho \text{ measured from inner edge of region}$	$\left. \begin{matrix} \cosh \mu(\rho - R_1) \\ \frac{\rho}{\sinh \mu(\rho - R_1)} \\ \rho \end{matrix} \right\} \rho \text{ measured from center of reactor}$	$\left. \begin{matrix} I_0(\mu\rho) \\ K_0(\mu\rho) \end{matrix} \right\} \rho \text{ measured from axis of reactor}$
A	ϕ_f^{i-1}	$R_1\phi_f^{i-1}$	$R_1 \left[\mu_f K_1(\mu_f R_1) \phi_f^{i-1} + \frac{J_f^{i-1}}{D_f} K_0(\mu_f R_1) \right]$
B	$\frac{J_f^{i-1}}{\mu_f D_f}$	$\frac{R_1 J_f^{i-1}}{\mu_f D_f} + \frac{\phi_f^{i-1}}{\mu_f}$	$R_1 \left[\mu_f I_1(\mu_f R_1) \phi_f^{i-1} - \frac{J_f^{i-1}}{D_f} I_0(\mu_f R_1) \right]$
C	$\phi_s^{i-1} - \frac{1}{S'}\phi_f^{i-1}$	$R_1 \left(\phi_s^{i-1} - \frac{1}{S'}\phi_f^{i-1} \right)$	$R_1 \left[\mu_s K_1(\mu_s R_1) \left(\phi_s^{i-1} - \frac{1}{S'}\phi_f^{i-1} \right) + \left(J_s^{i-1} - \frac{D_s}{S' D_f} J_f^{i-1} \right) \frac{K_0(\mu_s R_1)}{D_s} \right]$
E	$\frac{J_s^{i-1}}{\mu_s D_s} - \frac{J_f^{i-1}}{S' \mu_s D_f}$	$\frac{\phi_s^{i-1} - \frac{1}{S'}\phi_f^{i-1}}{\mu_s} + \frac{R_1}{\mu_s D_s} \left(J_s^{i-1} - \frac{D_s}{S' D_f} J_f^{i-1} \right)$	$R_1 \left[\mu_s I_1(\mu_s R_1) \left(\phi_s^{i-1} - \frac{1}{S'}\phi_f^{i-1} \right) - \left(J_s^{i-1} - \frac{D_s}{S' D_f} J_f^{i-1} \right) \frac{I_0(\mu_s R_1)}{D_s} \right]$

The outer values of fluxes and currents are obtained from the matrix product

$$\|Y\| \|\xi\| \text{ (previous region)} = \|\xi\| = \begin{bmatrix} \phi_f \\ \phi_s \\ J_f \\ J_s \end{bmatrix}$$

Outer Reflector:

$$\phi_f(\rho) = AX(\mu_f\rho)$$

$$\phi_s(\rho) = \frac{1}{S'} \phi_f(\rho) + BX(\mu_s\rho)$$

	Slab	Sphere	Cylinder
$X(\mu\rho)$	$\sinh \mu(T - \rho)$ ρ is measured from inner edge of region	$\frac{\sinh \mu(R_2 - \rho)}{\rho}$	$I_0(\mu\rho) - \frac{I_0(\mu R_2)}{K_0(\mu R_2)} K_0(\mu\rho)$
A	$\frac{\phi_f^{i-1}}{\sinh \mu_f T}$	$\frac{R_1 \phi_f^{i-1}}{\sinh \mu_f (R_2 - R_1)}$	$\frac{\phi_f^{i-1}}{I_0(\mu_f R_1) - [I_0(\mu_f R_2)/K_0(\mu_f R_2)]K_0(\mu_f R_1)}$
A (check)	$\frac{-J_f^{i-1}}{\mu_f D_f \cosh \mu_f T}$	$\frac{-(R_1/D_f)J_f^{i-1}}{(1/R_1) \sinh \mu_f (R_2 - R_1) + \mu_f \cosh \mu_f (R_2 - R_1)}$	$\frac{J_f^{i-1}}{\mu_f D_f I_1(\mu_f R_1) + [I_0(\mu_f R_2)/K_0(\mu_f R_2)]K_1(\mu_f R_2)}$
B	$\frac{\phi_s^{i-1} - (1/S')\phi_f^{i-1}}{\sinh \mu_s T}$	$\frac{\phi_s^{i-1} - (1/S')\phi_f^{i-1}}{(1/R_1) \sinh \mu_s (R_2 - R_1)}$	$\frac{\phi_s^{i-1} - (1/S')\phi_f^{i-1}}{I_0(\mu_s R_1) - [I_0(\mu_s R_2)/K_0(\mu_s R_2)]K_0(\mu_s R_1)}$
B (check)	$\frac{-J_s^{i-1} + (D_s/S'D_f)J_f^{i-1}}{\mu_s D_s \cosh \mu_s T}$	$\frac{(-R_1/D_s)J_s^{i-1} + (R_1/S'D_f)J_f^{i-1}}{(1/R_1) \sinh \mu_s (R_2 - R_1) + \mu_s \cosh \mu_s (R_2 - R_1)}$	$\frac{J_s^{i-1} - (D_s/S'D_f)J_f^{i-1}}{\mu_s D_s [I_1(\mu_s R_1) + [I_0(\mu_s R_2)/K_0(\mu_s R_2)]K_1(\mu_s R_1)]}$

9 EVALUATION OF MATERIAL CONSTANTS OF THE REACTOR

By Otto Schulze and J. R. Dietrich

In order to set up the reactor equations for a given thermal reactor (cf. Arts. 5 and 10), it is necessary to evaluate a number of constants which are characteristic of the materials of the reactor and their disposal in the reactor. In the reactor core, these constants are L^2 and D_s , which apply to the diffusion of thermal neutrons; k , the infinite multiplication constant, which is usually broken down as the product of four other constants η , ϵ , p , and f ; and constants characterizing the behavior of fast neutrons in the reactor. It will be assumed here that one of the simpler descriptions of the slowing-down distribution will be employed and that the behavior of fast neutrons can be characterized by two constants, a slowing-down area τ (or L_f^2) and a diffusion constant D_f . If a reflector is provided on the core, the same constants must be evaluated in the reflector also. Usually, k is zero in the reflector.

If the structure of the reactor is so fine-grained that none of its parts exceeds a small fraction of a diffusion length in its *smallest* dimension, the reactor may be considered to be effectively homogeneous and computation of the material constants is considerably simplified. If this condition is not met, the reactor is generally referred to as a *lumped* reactor.† The particular type of lumping, in which the fissionable material (fuel) is arranged in discrete lumps throughout the moderator—generally in a regular lattice pattern—is of frequent occurrence in reactor design. This type of arrangement reduces the absorption of resonance neutrons by U^{238} and thereby allows criticality to be attained with lower U^{235}/U^{238} ratios (lower enrichment) than would be possible with a homogeneous arrangement.

In the following articles, evaluation of the material constants will be discussed both for the effectively homogeneous case and for the lumped case described above.

9.1 The Infinite Multiplication Constant k

The infinite multiplication constant is arrived at in the following way: It is first assumed that the region for which k is to be evaluated extends in all directions to infinity. In this infinite medium, k is defined as the number of secondary neutrons reaching thermal energy per primary thermal neutron absorbed in the medium and can be broken down into the product of four terms:

$$k = \eta\epsilon pf \quad (168)$$

where η is called the regeneration factor, ϵ the fast fission factor, p the resonance escape probability, and f the thermal utilization. Equation (168) is often referred to as the four-factor formula. Evaluation of these factors is discussed in the four immediately following articles.

9.2 The Regeneration Factor η

The quantity η is defined as the number of fission neutrons produced per neutron absorbed in the fuel. The definition of fuel is to some extent arbitrary. In a reactor which is effectively homogeneous, the term fuel is usually applied to the fissionable isotope. In a lumped reactor the term usually refers to the fissionable isotope plus any isotopes which are mixed with the fissionable isotope or are geometrically closely associated with it. If the reactor is effectively homogeneous and the fuel is considered to consist of the fissionable isotope only, η is a constant characteristic of the fissionable isotope. For thermal neutrons, η is given for a number of fissionable isotopes in Table 7. If the fuel is lumped, it may consist of a number of isotopes, some of which are fissionable and some not. If the various isotopes are intimately

† The reactor might also be referred to, in terms of its neutron properties, as a heterogeneous reactor. This terminology leads to some confusion, however, as the terms homogeneous and heterogeneous have been used to characterize the state of actual physical homogeneity of the reactor.

mixed, they are all exposed to the same average neutron flux and η is given by

$$\eta = \frac{\sum_i \nu_i N_i \sigma_{fi}}{\sum_i N_i \sigma_{ai}} \quad (169)$$

where N_i is the number of atoms of the i th isotope per unit volume, σ_{fi} is the microscopic fission cross section of the i th isotope, ν_i is the average number of neutrons emitted per fission of the i th isotope, and σ_{ai} is the microscopic absorption cross section of the i th isotope.

In some cases it may be convenient to consider as part of the fuel nonfissionable isotopes which are closely associated with but not intimately mixed with the fissionable

Table 7. Number of Fission Neutrons Emitted per Thermal Neutron Absorbed (η) by Thermally Fissionable Isotopes*

Isotope	η
U ²³⁵	2.08 ± 0.02
U ²³³	2.31 ± 0.03
Pu ²³⁹	2.03 ± 0.03

* From "Neutron Cross Sections," McGraw-Hill Book Company, Inc., 1955.

isotope. An example is the material used for the cladding of fuel elements. In such a case the average flux may not be the same in all isotopes considered as part of the fuel. The quantity η is then given by

$$\eta = \frac{\sum_i \nu_i N_i \sigma_{fi} \bar{\phi}_i}{\sum_i N_i \sigma_{ai} \bar{\phi}_i} \quad (170)$$

where $\bar{\phi}$ is the average flux over the i th isotope. For symmetrical cases, $\bar{\phi}_i$ may be computed by methods similar to those discussed in Art. 9.3.

9.3 The Thermal Utilization f

The thermal utilization is defined as the ratio of the number of neutrons absorbed per unit time in fuel to the total number of neutrons absorbed per unit time in the reactor. For an effectively homogeneous reactor it is given by

$$f = \frac{\Sigma_a(\text{fuel})}{\Sigma_a(\text{fuel}) + \Sigma_a(\text{all other materials})} = \frac{1}{1 + [\Sigma_a(\text{all other materials})/\Sigma_a(\text{fuel})]} \quad (171)$$

where Σ_a is the macroscopic absorption cross section.

In a lumped reactor the absorption cross sections of the various materials must be weighted with the average thermal neutron flux over the materials. In computing f for such a case the reactor core material is usually considered to be infinite in extent. If the thermal neutron flux ϕ in such an infinite system is written as a function of some space coordinate r , then f is given by

$$f = \frac{\int \Sigma_{a1}(r) \phi(r) dV}{\int \Sigma_a(r) \phi(r) dV} \quad (172)$$

where the integration is taken over a typical volume of reactor material. The quantity Σ_{a1} is the macroscopic absorption cross section of the fuel material only, and Σ_a is the total macroscopic cross section, including both fuel and any other materials present.

If the reactor core is made up of regularly spaced regions of fuel and moderator, each of which can be considered individually as homogeneous, and if the macroscopic absorption cross section is designated by Σ_{a1} in the fuel and by Σ_{a2} in the moderator, then f is given by

$$f = \frac{\Sigma_{a1} \int_{\text{fuel}} \phi(r) dV}{\Sigma_{a1} \int_{\text{fuel}} \phi(r) dV + \Sigma_{a2} \int_{\text{moderator}} \phi(r) dV} \tag{173}$$

where the integrations are taken as indicated over a typical fuel region and a typical moderator region. It is convenient to work with the average fluxes in the fuel and moderator, $\bar{\phi}_1$ and $\bar{\phi}_2$, respectively:

$$\bar{\phi}_1 = \frac{1}{V_1} \int_{\text{fuel}} \phi(r) dV \tag{174}$$

and

$$\bar{\phi}_2 = \frac{1}{V_2} \int_{\text{moderator}} \phi(r) dV \tag{175}$$

where V_1 and V_2 are the volumes of a typical fuel region and a typical moderator region, respectively. Then

$$f = \frac{V_1 \Sigma_{a1} \bar{\phi}_1}{V_1 \Sigma_{a1} \bar{\phi}_1 + V_2 \Sigma_{a2} \bar{\phi}_2} = \frac{1}{1 + (\Sigma_{a2}/\Sigma_{a1})(V_2/V_1)(\bar{\phi}_2/\bar{\phi}_1)} \tag{176}$$

The ratio $\bar{\phi}_2/\bar{\phi}_1$ is called the disadvantage factor. $\bar{\phi}_2/\bar{\phi}_1$ is given explicitly in terms of f by

$$\text{Disadvantage factor} = \frac{\bar{\phi}_2}{\bar{\phi}_1} = \frac{\Sigma_{a1} V_1}{\Sigma_{a2} V_2} \left(\frac{1}{f} - 1 \right) \tag{177}$$

For a fuel-moderator lattice of the type considered here, the first step in calculating f is the division of the lattice into a number of identical unit cells in such a way that the unit cell possesses the maximum possible symmetry. Thus, if the fuel is disposed in equally spaced slabs, the cell breakdown is usually made as in Fig. 11.

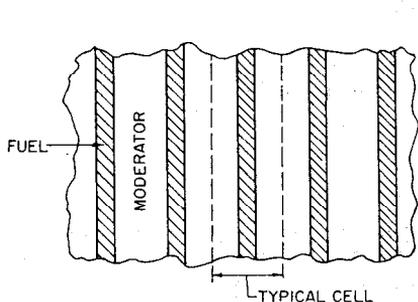


FIG. 11. Section of reactor with fuel in slabs.

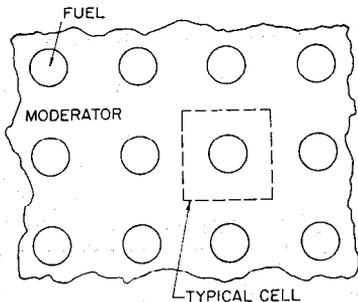


FIG. 12. Section of reactor with fuel in long round rods.

Figure 12 is a typical breakdown for a reactor using fuel rods in the form of long cylinders, disposed in a square array. The next step is to approximate the true shape of the cell by a shape which can be described by a single dimension. This must be done in such a way as to maintain the true volume of the cell: thus, the cell having the shape of a long square prism with side length as in Fig. 12 is approximated by a long circular cylinder of radius $R_2 = a/\sqrt{\pi}$ (Fig. 13).

With the problems reduced to one in a single dimension by a suitable cell approximation, f may be computed by diffusion theory or by higher order approximations to

transport theory. Only diffusion theory is treated here, although it cannot be considered a good approximation for the cases of small dimensions and strong absorption which often occur in the cell treatment.

The case of the cylindrical cell will be treated as typical (Fig. 13), and solutions will be given for other cases (Table 8).

The cell problem is treated as a problem in the diffusion of thermal neutrons; the slowing-down density is assumed to be uniform over the region of the cell occupied by moderator (region 2) and to be zero in the fuel (region 1). There is thus a source of q neutrons/(cm³)(sec) uniformly throughout the moderator, and the diffusion equation is

$$D_1 \nabla^2 \phi_1 - \Sigma_{a1} \phi_1 = 0 \quad \text{in the fuel} \quad (178)$$

$$D_2 \nabla^2 \phi_2 - \Sigma_{a2} \phi_2 + q = 0 \quad \text{in the moderator} \quad (179)$$

where D is the diffusion constant and Σ_a the macroscopic absorption cross section. Writing $\kappa = \Sigma_a/D$, the solutions of Eqs. (177) and (178) are, respectively,

$$\phi_1 = AI_0(\kappa_1 r) \quad (180)$$

$$\phi_2 = BI_0(\kappa_2 r) + CK_0(\kappa_2 r) + \frac{q}{\Sigma_{a2}} \quad (181)$$

where the K_0 solution in region 1 has been precluded by the requirement that ϕ be finite at $r = 0$.

FIG. 13. Square cell of Fig. 12 cylindricalized.

Since all the cells are identical and are assumed to constitute an infinite array, there must be no neutron current from cell to cell; hence $d\phi_2(R_2)/dr = 0$. Application of this boundary condition to Eq. (181) gives the relation between the constants B and C :

$$C = \frac{I_1(\kappa_2 R_2)}{K_1(\kappa_2 R_2)} B \quad (182)$$

and Eq. (181) can be rewritten as

$$\phi_2 = G[K_1(\kappa_2 R_2)I_0(\kappa_2 r) + I_1(\kappa_2 R_2)K_0(\kappa_2 r)] + \frac{q}{\Sigma_{a2}} \quad (183)$$

where G is a new constant

$$G = \frac{B}{K_1(\kappa_2 R_2)} \quad (184)$$

Table 8. Values of E and F for Cylindrical, Slab, and Spherical Geometry*

Cell geometry	F	E
Cylinder	$\frac{\kappa_1 R_1}{2} \frac{I_0(\kappa_1 R_1)}{I_1(\kappa_1 R_1)}$	$\frac{\kappa_2(R_2^2 - R_1^2)}{2R_1} \left[\frac{I_0(\kappa_2 R_1)K_1(\kappa_2 R_2) + K_0(\kappa_2 R_1)I_1(\kappa_2 R_2)}{I_1(\kappa_2 R_2)K_1(\kappa_2 R_1) - K_1(\kappa_2 R_2)I_1(\kappa_2 R_1)} \right]$
Slab	$\kappa_1 R_1 \coth \kappa_1 R_1$	$\kappa_2(R_2 - R_1) \coth \kappa_2(R_2 - R_1)$
Sphere	$\frac{\kappa_1^2 R_1^2}{3} \frac{\tanh \kappa_1 R_1}{\kappa_1 R_1 - \tanh \kappa_1 R_1}$	$\frac{\kappa_2^2(R_2^3 - R_1^3)}{3R_1} \left[\frac{1 - \kappa_2 R_2 \coth \kappa_2(R_2 - R_1)}{1 - \kappa_2 R_2 R_1 - \kappa_2(R_2 - R_1) \coth \kappa_2(R_2 - R_1)} \right]$

For sphere and cylinder, R_1 is the radius of fuel, R_2 the cell radius. For slab, R_1 is the half thickness of fuel, R_2 the half thickness of the cell. Subscripts 1 refer to fuel and 2 to moderator in all cases.

* A. M. Weinberg, "Science and Engineering of Nuclear Power," vol. II, chap. 6, Addison-Wesley Publishing Company, Reading, Mass., 1949.

Applying the conditions $\phi_1(R_1) = \phi_2(R_1)$ and $D_1\phi_1'(R_1) = D_2\phi_2'(R_1)$ at the interface between regions 1 and 2 to Eqs. (180) and (183) yields the values of the constants A and G :

$$\frac{1}{A} = \frac{\Sigma_{a2}}{q} \left\{ I_0(\kappa_1 R_1) - \frac{D_1 \kappa_1}{D_2 \kappa_2} \frac{I_1(\kappa_1 R_1) [K_1(\kappa_2 R_2) I_0(\kappa_2 R_1) + I_1(\kappa_2 R_2) K_0(\kappa_2 R_1)]}{K_1(\kappa_2 R_2) I_1(\kappa_2 R_1) - I_1(\kappa_2 R_2) K_1(\kappa_2 R_1)} \right\} \quad (185)$$

$$\text{and} \quad \frac{1}{G} = \frac{1}{A} \frac{D_2 \kappa_2}{D_1 \kappa_1} \frac{K_1(\kappa_2 R_2) I_1(\kappa_2 R_1) - I_1(\kappa_2 R_2) K_1(\kappa_2 R_1)}{I_1(\kappa_1 R_1)} \quad (186)$$

although, for the evaluation of f , the value of G is not required.

To evaluate f the basic definition of f is recalled:

$$f = \frac{\text{absorption rate in fuel}}{\text{total absorption rate}}$$

These equations will be evaluated for unit length of cell. The absorption rate in fuel per unit length is just the net neutron current into region 1 per unit length, or

$$\begin{aligned} \text{Absorption by fuel per unit length} &= 2\pi R_1 J(R_1) = 2\pi R_1 \left[D_1 \frac{d\phi_1}{dr}(R_1) \right] \\ &= 2\pi R_1 A \frac{\Sigma_{a1}}{\kappa_1} I_1(\kappa_1 R_2) \end{aligned}$$

The total absorption per unit length of cell is just the total source strength per unit length:

$$\text{Total absorption per unit length} = \pi(R_2^2 - R_1^2)q$$

whence

$$\frac{1}{f} = \frac{\pi q \kappa_1 (R_2^2 - R_1^2)}{2\pi R_1 \Sigma_{a1} I_1(\kappa_1 R_1)} \frac{1}{A} = \frac{q \kappa_1 (R_2^2 - R_1^2)}{2 \Sigma_{a1} R_1 I_1(\kappa_1 R_1)} \frac{1}{A} \quad (187)$$

and, inserting the value for $1/A$,

$$\frac{1}{f} = \frac{V_2 \Sigma_{a2}}{V_1 \Sigma_{a1}} \left[\frac{\kappa_1 R_1}{2} \frac{I_0(\kappa_1 R_1)}{I_1(\kappa_1 R_1)} \right] + \kappa_2 \frac{(R_2^2 - R_1^2)}{2R_1} \left[\frac{I_0(\kappa_2 R_1) K_1(\kappa_2 R_2) + K_0(\kappa_2 R_1) I_1(\kappa_2 R_2)}{I_1(\kappa_2 R_2) K_1(\kappa_2 R_1) - K_1(\kappa_2 R_2) I_1(\kappa_2 R_1)} \right] \quad (188)$$

Thus $1/f$ consists of two terms, one of which involves κ of the fuel only and the other κ of the moderator only. This form of expression for $1/f$ can be derived for other one-dimensional geometries, the customary form of the equation being

$$\frac{1}{f} = 1 + \frac{V_2 \Sigma_{a2}}{V_1 \Sigma_{a1}} F + (E - 1) \quad (189)$$

Here F , a function of κ_1 and R_1 only, is actually the ratio of the flux density at the fuel-moderator interface to the average flux density in the fuel. The quantity $(V_2 \Sigma_{a2} / V_1 \Sigma_{a1}) F$ is often called the relative absorption. The quantity $E - 1$ is called the excess absorption. It accounts for the effect of the higher average flux density in the moderator relative to that at the interface. The values of F and E for the case of cylindrical symmetry are obvious by comparison of Eqs. (188) and (189). They are repeated in Table 8, along with the values for the cases of spherical geometry and slab geometry.

If the fuel is surrounded by a thin† layer of nonfissionable material of volume V_3 , per unit length of cell and macroscopic absorption cross section Σ_{a3} , its effect can be included by modifying the equation for $1/f$ to

$$\frac{1}{f} = 1 + \left(\frac{V_2 \Sigma_{a2} + V_3 \Sigma_{a3}}{V_1 \Sigma_{a1}} \right) F + (E - 1) \quad (190)$$

where the values of F and E remain unchanged.

† "Thin" implying that the effect of the layer on the flux distribution—whether the effect be due to absorption, diffusion constant, or slowing-down properties—is negligible.

9.4 The Resonance Escape Probability p †

In the expression for k of a thermal reactor the resonance escape probability of interest is that over the neutron energy range from the upper energy limit of important resonance absorption to thermal energy. If the upper energy limit is designated E_0 , then the resonance escape probability to thermal energy $p(th)$ for an effectively homogeneous reactor containing a resonance absorber is

$$p(th) = \exp \left\{ - \int_{E_{th}}^{E_0} \frac{\Sigma_a(E')}{\bar{\xi}(E')[\Sigma_s(E') + \Sigma_a(E')]} \frac{dE'}{E'} \right\} \quad (191)$$

where $\Sigma_a(E')$ is the macroscopic absorption cross section, $\Sigma_s(E')$ the macroscopic scattering cross section, and $\bar{\xi}(E')$ the average logarithmic energy decrement per collision (Art. 3.1) in the medium. These quantities are seldom known in sufficient detail to permit evaluation of Eq. (191); an experimentally determined effective resonance integral $\int_{E_{th}}^{E_0} \frac{\sigma_{aeff}}{E'} dE'$ is usually used for evaluation of p . If $\bar{\xi}$ and Σ_s can be considered constant with energy (usually true in the resonance range), $p(th)$ is given by

$$p(th) = \exp \left(- \frac{1}{\bar{\xi}} \frac{N_a}{\Sigma_s} \int_{E_{th}}^{E_0} \sigma_{aeff} \frac{dE'}{E'} \right) \quad (192)$$

where N_a is the number of absorber atoms per unit volume. Although the effective resonance integral is an experimentally determined quantity, it is, in principle, given by

$$\int_{E_{th}}^{E_0} \sigma_{aeff} \frac{dE'}{E'} = \int_{E_{th}}^{E_0} \frac{1}{1 + (N_a \sigma_a / \Sigma_s)} \sigma_a \frac{dE'}{E'} \quad (193)$$

and is a function only of the species of absorber and the ratio N_a/Σ_s . Figure 14 gives experimentally determined curves of the effective resonance integrals as functions of N_a/Σ_s for uranium and thorium, the two most important resonance absorbers which occur in reactor problems. The procedure for computing $p(th)$ for these cases is as follows: (1) Compute N_a/Σ_s , where Σ_s is the total macroscopic scattering cross section in the medium,¹ including the contributions of all scatterers present. (2) Find the corresponding value of $\int_{E_{th}}^{E_0} \sigma_{aeff} \frac{dE'}{E'}$ from the curve Fig. 14. (3) Use this value in Eq. (192) to compute $p(th)$, employing a value of $\bar{\xi}$ consistent with the scattering cross sections previously used [Eq. (37)].

If lumped resonance absorbers are present, the resonance absorption density inside a lump is much lower than that at the surface because of the very strong absorption by the resonances. If the material of the lump has very low moderating power, the energy regions depleted of flux by the strong resonance absorption near the surface are not replenished by moderation, and this "self-protection" of the absorber against resonance neutron absorption is particularly great. It is this effect which makes possible the construction of a critical reactor from lumps of natural uranium in a moderator, despite the high resonance cross sections of U^{238} which, in a uniform mixture of uranium and moderator, would absorb too many neutrons to permit criticality.

Unless the lumped resonance absorber has a simple and well-known resonance structure, its absorption must be determined by a semiempirical method. The case of uranium is typical of the latter approach.

It is predicted theoretically and is found experimentally to be true that the rate of resonance absorption by lumps of uranium of simple shape (no. reentrant surfaces) can be represented by an expression of the form

$$A(E) = \bar{\phi}_1(E) N_1 V_1 a(E) + \bar{\phi}_s(E) N_1 V_1 b(E) \frac{S}{M} \quad (194)$$

† Cf. Art. 3.5.

where $A(E)$ is the number of resonance absorptions per second per unit energy interval, $\bar{\phi}_1(E)$ is the average flux per unit energy interval in the interior of the lump, $\bar{\phi}_s(E)$ is the average flux per unit energy interval over the surface of the lump, N_1 is the density of absorbing atoms in the lump, and V_1 , S , and M are the volume, surface area, and mass of the lump, respectively; $a(E)$ and $b(E)$ are to be determined experimentally.

If the lumped resonance absorbers are arranged in a regular lattice, some volume of moderator V_2 will be associated with each fuel lump, the absorber plus moderator making up a cell similar to those previously discussed (Figs. 11 to 13). It is assumed

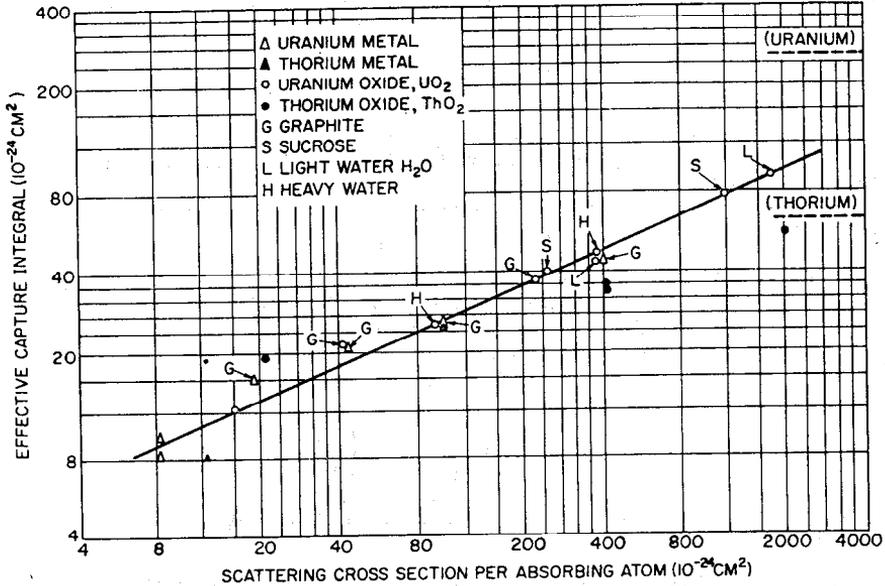


FIG. 14. Effective resonance-capture integrals of U²³⁸ and Th²³² diluted with neutron-scattering materials. (Reproduced from R. L. Macklin and H. S. Pomerance, *Resonance Capture Integrals, Geneva Conf. Paper A/Conf.8/P.833/Rev.1.*)

that there is no resonance absorption in the moderator. The decrease with decreasing energy of the number of neutrons slowing down in the cell past any energy E is then due to the absorption in the lump; i.e.,

$$A(E) = \frac{dQ(E)}{dE} \tag{195}$$

and $Q(E)$ is given, very nearly [Eq. (41)], by

$$Q(E) = V_2 \bar{\phi}_2(E) \bar{\xi}_2 \Sigma_{s2} E \tag{196}$$

whence, combining Eqs. (194), (195), and (196),

$$\frac{1}{Q} \frac{dQ}{dE} = \frac{1}{E} \left[\frac{\bar{\phi}_1(E) N_1 V_1 a(E)}{V_2 \bar{\phi}_2(E) \bar{\xi}_2 \Sigma_{s2}} + \frac{\bar{\phi}_s(E) N_1 V_1 b(E) S/M}{V_2 \bar{\phi}_2(E) \bar{\xi}_2 \Sigma_{s2}} \right] \tag{197}$$

and integrating between the limits E_{th} and E_0 , $p(th)$ is given as

$$p(th) = \frac{Q(E_{th})}{Q(E_0)} = \exp \left\{ - \frac{N_1 V_1}{V_2 \bar{\xi}_2 \Sigma_{s2}} \left[\int_{E_{th}}^{E_0} \frac{\bar{\phi}_1(E')}{\bar{\phi}_2(E')} a(E') \frac{dE'}{E'} + \frac{S}{M} \int_{E_{th}}^{E_0} \frac{\bar{\phi}_s(E')}{\bar{\phi}_2(E')} b(E') \frac{dE'}{E'} \right] \right\} \tag{198}$$

If it is assumed that $\bar{\phi}_s(E)/\bar{\phi}_2(E) = \bar{\phi}_1(E)/\bar{\phi}_2(E)$ and that the ratio $\bar{\phi}_1(E)/\bar{\phi}_2(E)$ is independent of energy in the energy range of important resonance absorption, then

$$\begin{aligned} p(th) &= \exp \left[-\frac{N_1 V_1}{V_2 \bar{\xi}_2 \Sigma_{s2}} \frac{\bar{\phi}_1}{\bar{\phi}_2} \int_{E_{th}}^{E_0} a(E') \frac{dE'}{E'} + \frac{S}{M} b(E') \frac{dE'}{E'} \right] \\ &= \exp \left[-\frac{N_1 V_1}{V_2 \bar{\xi}_2 \Sigma_{s2}} \frac{\bar{\phi}_1}{\bar{\phi}_2} \int_{E_{th}}^{E_0} \sigma_{a,eff}(E') \frac{dE'}{E'} \right] \end{aligned} \quad (199)$$

The above argument makes it appear reasonable that the effective resonance integral for a lumped resonance absorber in a lattice arrangement can be represented, for various geometries of absorber, by an expression of the form

$$\int_{\text{resonance}} \sigma_{a,eff}(E') \frac{dE'}{E'} = A + \mu \frac{S}{M} \quad (200)$$

Where the subscript "resonance" on the integral sign denotes that the integration is taken over that energy range which includes all important resonances, from thermal to fission energy.

Values of A and μ which have been used for natural uranium and natural uranium compounds are given in Table 9.

Table 9. Resonance Constants for Natural Uranium Metal and Some of Its Compounds, for Use with Eq. (200)*

Material	A , barns	μ , barn g/cm ²
U	9.25	24.7
U ₂ O ₈	12	20
UO ₂	11.51	22.1
UF ₆	14.6	16.3

* From "The Reactor Handbook," vol. 1, Table 1.5.26, AECD 3645, March, 1955. See original table for sources of information.

In order to compute the resonance escape probability, it is necessary to evaluate the ratio $\bar{\phi}_1/\bar{\phi}_2$ [Eq. (199)]. The computation of this ratio is formally identical with the computation of the disadvantage factor (Art. 9.3). If a resonance disadvantage factor f_r is defined in a manner completely analogous to the disadvantage factor of Art. 9.3, then [cf. Eq. (177)]

$$\begin{aligned} \frac{\bar{\phi}_1}{\bar{\phi}_2} &= \frac{1}{\text{resonance disadvantage factor}} = \frac{\Sigma_{a2} V_2}{\Sigma_{a1} V_1} \frac{1}{1/f_r - 1} \\ &= \frac{1}{F + (\Sigma_{a1} V_1 / \Sigma_{a2} V_2)(E - 1)} \end{aligned} \quad (201)$$

[see Eq. (189)], and

$$p(th) = \exp \left[-\frac{1}{\frac{V_2 \bar{\xi}_2 \Sigma_{s2} F}{N_1 V_1 \int_{\text{resonance}} \sigma_{a,eff}(E') (dE'/E')} + \frac{\Sigma_{a1} \bar{\xi}_2 \Sigma_{s2} (E - 1)}{\Sigma_{a2} N_1 \int_{\text{resonance}} \sigma_{a,eff}(E') (dE'/E')}} \right] \quad (202)$$

In this expression the symbol Σ_{a2} denotes a fictitious "absorption" cross section (macroscopic) which is really a slowing-down cross section, characterizing the probability that a neutron will be slowed out of the energy range occupied by the resonances. It is equal to Σ_s , the number of scattering collisions per unit volume per second per unit flux divided by $1/\bar{\xi} \ln E_1/E_2$, the average number of collisions required

to slow a neutron through the resonance region covering the energy range E_1 to E_2 :

$$\Sigma_{a2} = \frac{\Sigma_{s2}}{(1/\xi_2) \ln (E_1/E_2)} \quad (203)$$

The average absorption cross section in the absorber Σ_{a1} is, by definition,

$$\Sigma_{a1} = \frac{N_1 \int_{\text{resonance}} \sigma_{a,eff}(E')(dE'/E')}{\int_{\text{resonance}} dE'/E'} = \frac{N_1 \int_{\text{resonance}} \sigma_{a,eff}(E')(dE'/E')}{\ln (E_1/E_2)} \quad (204)$$

From Eqs. (203) and (204),

$$\frac{\Sigma_{a1}}{\Sigma_{a2}} = \frac{N_1}{\xi_2 \Sigma_{s2}} \int_{\text{resonance}} \sigma_{a,eff}(E') \frac{dE'}{E'} \quad (205)$$

and Eq. (202) becomes

$$p(th) = \exp \left[- \frac{1}{\frac{V_2 \xi_2 \Sigma_{s2} F}{N_1 V_1 \int_{\text{resonance}} \sigma_{a,eff}(E')(dE'/E')} + (E - 1)} \right] \quad (206)$$

In order to apply Eq. (206), it is necessary to know the effective values of κ for the absorber and the moderator (for evaluation of E and F) and an effective value of $\xi_2 \Sigma_{s2}$ for the moderator. Experimentally determined values of κ_1 are given for uranium and uranium oxide in Table 10, and values of κ_2 and $\xi_2 \Sigma_{s2}$ are given for various moderators in Table 11.

The procedure for calculating the resonance escape probability is, then, the following. (1) Set up a cell system appropriate to the lattice geometry (cf. Figs. 11 to 13). Designate the fuel as region 1 and the moderator as region 2. (2) Take values of κ_1 and κ_2 from Tables 10 and 11 and use them with the cell dimensions to calculate F

Table 10. Values of κ_1 at Resonance Energy in Natural Uranium Metal and U_3O_8

U	$\kappa_1 = 0.0222\rho$	cm ⁻¹
U_3O_8	$\kappa_1 = 0.025\rho$	cm ⁻¹

ρ = density, g/cm³.

Table 11. Moderator Resonance Constants*

Material	$\overline{\xi\sigma_s}$, barns per atom or per molecule	κ_2/ρ , cm ² /g
H ₂ O	38.5	0.583
D ₂ O	5.28	0.141
Be	1.26	0.128
BeO	1.76	0.0690
Graphite	0.76	0.0672

* From "The Reactor Handbook," vol. 1, Table I.5.27. (See Ref. 17.) Original source CL-697, IV E.

Values of κ_2/ρ are for use with U; if fuel is oxide, multiply by 0.88.

ρ = density, g/cm³.

and $(E - 1)$ by the appropriate equations of Table 8. (3) Compute the surface/mass ratio (S/M) of the fuel lump, and use it, with values of A and μ from Table 9, to compute the effective resonance integral $\int_{\text{resonance}} \sigma_{a,eff}(E')(dE'/E')$ by Eq. (200). (4) Take the effective value of $\xi_2 \Sigma_{s2}$ for the moderator from Table 11, and combine with the other quantities determined above to find $p(th)$ by Eq. (206).

9.5 The Fast Fission Effect ϵ

The fast fission effect applies to the fission of U^{238} nuclei by fast neutrons in reactors containing relatively large proportions of U^{238} to U^{235} . It is usually a negligible effect in effectively homogeneous thermal reactors because of the usually low ratio of U^{238} to moderator in such reactors. Only neutrons having energies above about 1 Mev can cause fission of U^{238} . For this reason the following two assumptions can often be made in calculating ϵ for lumped reactors:

1. Any inelastic scattering collision within the fuel reduces the neutron energy below the U^{238} fission threshold.

2. Any neutron which escapes the fuel lump is degraded to an energy below the U^{238} fission threshold by the moderator before it reaches a new fuel lump or is scattered back into the original lump.

The quantity ϵ is defined as the number of neutrons slowing down past the fission threshold of U^{238} per neutron produced by thermal fission. With the above assumptions it is given by

$$\epsilon - 1 = \frac{[\nu - 1 - (\Sigma_c/\Sigma_f)](\Sigma_f/\Sigma_t)P}{1 - (\nu\Sigma_f + \Sigma_a)P'/\Sigma_t} \quad (207)$$

where Σ_c , Σ_f , Σ_a , and Σ_t are, respectively, appropriate average values of the capture, fission, elastic scattering, and total cross sections in the fuel lump and ν is the average number of fast neutrons produced per fission of U^{238} by fast neutrons.

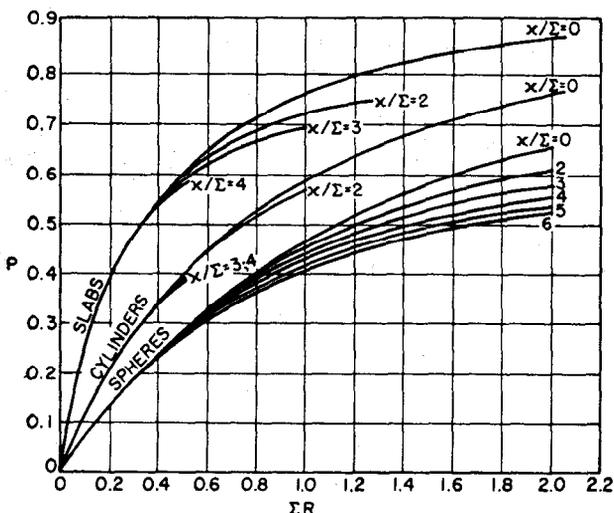


FIG. 15. Probability of first-flight collision within a fuel lump. Σ is the macroscopic total cross section, x the reciprocal of thermal diffusion length. For spheres and cylinders R is the radius; for slabs it is the half-thickness. P' is given by the curves for $x/\Sigma = 0$. (Reproduced from Fig. 15.14, *The Reactor Handbook*, vol. 1, AECD 3645, March, 1955; original from CD-644.)

In Eq. (207) P is the probability that a neutron produced by thermal fission in the fuel lump will make some kind of a collision within the lump before escaping. The neutrons produced within the lump by fast fission may, in turn, produce further fast fissions within the same lump, and on occasion a sequence of such fast fissions may last through several fast-neutron generations. The denominator of Eq. (207) takes account of these fission sequences. The factor P' is the probability that a neutron produced by fast fission in the fuel lump will make some kind of collision within the lump before escaping. In computing P it is assumed that the sources of fission neu-

trons under consideration are distributed in the fuel lump according to the thermal flux distribution in the lump, whereas in computing P' it is assumed that the sources are distributed uniformly within the lump. Computed curves of values of P for various lump sizes and geometries are given in Fig. 15. Note that P , since it involves the distribution of thermal fissions, is a function of both κ , the reciprocal of the thermal diffusion length in the lump, and $\Sigma_t R$, the dimension of lump measured in total mean free paths. The quantity P' for a given geometry is a function of $\Sigma_t R$ only; it is given in Fig. 15 by the curves for which $\kappa/\Sigma_t = 0$.

Values of the other constants needed for Eq. (207), for natural uranium metal, are given in Table 12.

Table 12. Fast Effect in Uranium

Constants for computation when there is no fast-neutron interaction between fuel lumps;*

Constant	
σ_f	0.29 barns
σ_{el}	1.5 "
σ_a	<0.04 "
σ_t	4.3 "
ν	<2.55 (dimensionless)

Experimental values of fast effect (ϵ) for 0.600-in.-diameter rods of slightly enriched uranium in light water. There is fast-neutron interaction between fuel lumps.†

Volume water Volume uranium	% U ²³⁵ in uranium (by weight)	ϵ
0.1785	1.299	1.227 ± 0.011
1	1.299	1.105 ± 0.002
1.5	1.299	1.072 ± 0.001
2	1.299	1.061 ± 0.001
3	1.299	1.047 ± 0.001
4	1.299	1.043 ± 0.001
1	1.143	1.109 ± 0.002
1.5	1.143	1.074 ± 0.001
4	1.143	1.042 ± 0.001

* From "The Reactor Handbook," vol. 1, Table 1.5.20, AECD-3645, March, 1955. Original source CL-697.

† H. Kouts, G. Price, K. Downes, R. Sher, and V. Walsh, Exponential Experiments with Slightly Enriched Uranium Rods in Ordinary Water, Geneva Conf. Paper A/Conf. 8/P/600, June 30, 1955.

In close-packed lattices such as those which occur in some water-moderated reactors, assumption 2 above does not hold, and there is a substantial increase of ϵ over that calculated by the foregoing method as a result of the mutual fast-neutron interactions between separate fuel lumps. Measured values of ϵ for some H₂O-uranium lattices of this type are given in Table 12.

9.6 The Thermal Diffusion Coefficient D_{th}

For an effectively homogeneous medium the thermal diffusion coefficient can be computed from Eq. (10) or, if the medium absorbs strongly, by Eq. (9). Measured values of λ_{tr} should be used if they are available. If they are not, the scattering cross sections must be used in conjunction with a value of $\bar{\mu}_0$ computed from Eq. (11). Table 13 gives transport mean free paths for some materials of importance. If the medium is a mixture of materials, a macroscopic transport cross section ($\Sigma_{tr} = 1/\lambda_{tr}$) can be obtained for each component and the total macroscopic transport cross section of the medium can be taken as their sum.

In the case of lumped reactors, the ratio of moderator volume to fuel volume is often sufficiently large that the moderator diffusion coefficient can be used as the diffusion coefficient of the medium. If this condition does not hold, an approximate value for an effective transport cross section can be computed by taking a flux-weighted average of the transport cross section over a lattice cell, exactly as is done for

the absorption cross section in Eq. (209). The quantity D_{th} can then be computed as for the homogeneous case above. The resulting expression for the effective diffusion coefficient is

$$\bar{D}_{th} = \frac{1}{3\Sigma_{tr}} = \frac{1}{3} \frac{1 + (V_1/V_2)(\bar{\phi}_1/\bar{\phi}_2)}{\Sigma_{tr2} + (V_1/V_2)(\bar{\phi}_1/\bar{\phi}_2)\Sigma_{tr1}}$$

where

$$\frac{V_1\bar{\phi}_1}{V_2\bar{\phi}_2} = \frac{f}{1-f} \frac{\Sigma_{a2}}{\Sigma_{a1}}$$

The approximation is a poor one for use in matching neutron currents at a reactor boundary, since the cell approximation does not hold at the boundaries. Such defects are inherent in the cell method and are the price paid for the simplicity of the method.

Table 13. Thermal Diffusion Length and Transport Mean Free Path of Various Materials*

Substance	Thermal diffusion length L , cm	Transport mean free path (thermal), cm
H ₂ O	2.85 ± 0.03	0.48 ± 0.01
D ₂ O (0.16% H ₂ O)	116 ± 4	2.65 ± 0.15
Be ($\rho = 1.85$)	20.8 ± 0.5	1.43 ± 0.05 (using $\sigma_{a(th)} = 9$ millibarns)
BeO ($\rho = 2.69$)	29 ± 2	0.90 (calculated from original reference—CP-3647, 1946)
Graphite (corrected to $\rho = 1.60$): GBF	54.4 ± 0.5 ($\sigma_{abs} = 4.4$ millibarns)†	
AGOT	52 ± 1 ($\sigma_{abs} = 4.8$ millibarns)†	
Th ($\rho = 11.2$)	2.7 ± 0.3	
ThO ₂ ($\rho = 6$)	4.1 ± 0.4	
U ($\rho = 18.9$)	1.55 ± 0.05	
U ₃ O ₈ ($\rho = 6.0$)	3.7 ± 0.4	

* From "The Reactor Handbook," Table 1.5.4, AECD-3645, March, 1955. See original table for sources of information.

† 2,200-m/sec values; obtained by using a calculated value of 4.70 barns for the Maxwellian average of the total cross section.

9.7 The Thermal Diffusion Area $L^2 (= 1/\kappa^2)$

The thermal diffusion area is given, in all cases, by the equation

$$L^2 = \frac{D_{th}}{\Sigma_{ath}} \quad (208)$$

where Σ_{ath} is the total thermal macroscopic cross section of the medium. In the case of a lumped reactor an effective absorption cross section $\bar{\Sigma}_a$ must be used in the reactor core. The effective value is taken as the volume average of the true absorption cross section, weighted by the thermal flux density; i.e., for a cell (see Figs. 11 to 13) where the fuel is designated by the subscript 1 and the moderator by the subscript 2

$$\begin{aligned} \bar{\Sigma}_a &= \frac{V_2\Sigma_{a2} + V_1\Sigma_{a1}(\bar{\phi}_1/\bar{\phi}_2)}{V_2 + V_1(\bar{\phi}_1/\bar{\phi}_2)} \\ &= \frac{\Sigma_{a2}}{1-f} \frac{V_2}{V_2 + V_1(\bar{\phi}_1/\bar{\phi}_2)} \end{aligned} \quad (209)$$

where $\bar{\phi}_1/\bar{\phi}_2$ is the reciprocal of the disadvantage factor [Eq. (177)] and f is the thermal utilization [Eq. (189)]. Usually the term $V_1(\bar{\phi}_1/\bar{\phi}_2)$ in Eq. (209) can be neglected in comparison with V_2 . For this case

$$\bar{\Sigma}_a = \frac{\Sigma_{a2}}{1-f} \quad (210)$$

and if the diffusion coefficient of the medium is taken to be that of the moderator, the diffusion area becomes, approximately,

$$L^2 = L_2^2(1 - f) \quad (211)$$

where L_2^2 is the diffusion area of the moderator.

Experimental values of thermal diffusion length for a number of materials are given in Table 13.

9.8 The Fast-diffusion Coefficient D_f

An effective fast-diffusion coefficient covering a range of energies can be defined only within the formalism of the group treatment of slowing down.

Most reactors can be considered effectively homogeneous with respect to fast neutrons, for even in lumped reactors the slowing-down lengths and absorption mean free paths are sufficiently long that large variations in fast flux do not occur over a typical lattice cell. Hence, at a given energy, a total transport cross section $\Sigma_{tr}(E)$ can be evaluated which is just the sum of the macroscopic transport cross sections of all the reactor core materials, considered to be uniformly mixed together. Thus a diffusion coefficient $D(E) = 1/3\Sigma_{tr}(E)$ can be evaluated as a function of energy over the energy range E_1 to E_2 covered by the fast-neutron group in question. Usually $D(E)$ will not vary rapidly with neutron energy, and it is therefore permissible to make the assumption that the variation in energy spectrum from point to point in the medium can be neglected. An effective diffusion coefficient D_f can then be evaluated for the group simply by averaging $D_f(E)$ over the energy distribution of flux $\phi(E)$:

$$D_f = \frac{\int_{E_1}^{E_2} D(E)\phi(E) dE}{\int_{E_1}^{E_2} \phi(E) dE} \quad (212)$$

If the energy distribution of flux can be represented by the $1/E$ distribution (Art. 3.2) the expression becomes

$$D_f = \frac{\int_{E_1}^{E_2} D(E)(dE/E)}{\ln E_2/E_1} \quad (213)$$

9.9 The Slowing-down Area (τ or L_f^2)

The computation of slowing-down area is quite difficult except in those cases for which the Fermi age approximation applies (the approximation applies fairly well for beryllium and moderators of higher mass number provided inelastic scattering is not important). When the approximation does apply, τ can be calculated by the equations of Art. 3.4. For other cases, see Ref. 5.

In all cases a measured value of τ should be used if one is available. Measured values are given for a number of moderators in Table 14. The table contains measured values for fission neutrons slowed down to the indium resonance energy and calculated values from the indium resonance energy to thermal. The total τ from fission to thermal energy is the sum of the two.

As pointed out above (Art. 9.8), even lumped reactors can usually be considered effectively homogeneous in computing fast neutron constants. In many lumped reactors the slowing-down area for the uranium-moderator lattice is very nearly the same as that for the moderator alone; the high inelastic scattering cross section of the uranium compensates for its low elastic slowing-down power.

In two-group calculations the slowing-down area is designated by L_f^2 instead of τ and is treated as a constant which may be adjusted to compensate for the deficiencies of the exponential description of slowing down distribution. These adjustments are discussed in Art. 5.6.

Table 14. Age of Fission Neutrons in Various Moderators*
(Room temperature)

Moderator	Age to indium resonance, cm ²	Calculated age indium resonance to thermal, cm ²	Age to thermal, cm ²
H ₂ O	30.4 ± 0.4	1	31.4
D ₂ O (0.16% H ₂ O)	100 ± 5		125
Be (ρ = 1.85)	80 ± 2	17.2	97.2
BeO (ρ = 3.0)			105 ± 10
C (ρ = 1.60)	311 ± 3	53	364
66.7% H ₂ O, 33.3% Al	49.6	2	51.6
50% H ₂ O, 50% Al	76.8	3	79.8
80% H ₂ O, 20% Zr	35.7		
50% H ₂ O, 50% Zr			
			61†
			(from reactor analysis)
D ₂ O (ρ = 1.1 g/cm ³)			127
D ₂ O + 1.4 g ThO ₂ /g D ₂ O			132 calc.‡
D ₂ O + 2.8 g ThO ₂ /g D ₂ O			140
(ThO ₂ density = 9.69 g/cm ³)			
96.7% D ₂ O, 3.3% U, by vol			107 ± 5§

Values of τ for H₂O-U mixtures can be inferred from M^2 measurements in the following reports: S. Krasik and A. Radkowski, Pressurized Water Reactor (PWR) Critical Experiments, *Geneva Conference Paper A/Conf. 8/P/601*, June 30, 1955; H. Kouts *et al.*, Exponential Experiments with Slightly Enriched Uranium Rods in Ordinary Water, *Geneva Conf. Paper A/Conf. 8/P/600*, June 30, 1955.

* The first eight values are from "The Reactor Handbook," Tables 1.5.1 and 1.5.2, AECD-3645, March, 1955. See original table for sources of information.

† J. Ernest Wilkins, Jr., Robert L. Hellens, and Paul F. Zweifel, Status of Experimental and Theoretical Information on Neutron Slowing Down Distributions in Hydrogenous Media, *Geneva Conf. Paper A/Conf. 8/P/597*, June, 1955.

‡ M. Tobias, Certain Nuclear Data and Physical Properties to Be Used in the Study of Thorium Breeders, CF-54-8-179, Aug. 26, 1954.

§ E. Richard Cohen, Exponential Experiments on D₂O-Uranium Lattices, *Geneva Conf. Paper A/Conf. 8/P/605*, June 30, 1955.

10 MULTIGROUP CALCULATIONS

By David Okrent

In Art. 5.8 the multigroup formulation was given for the case in which fission occurs only in the thermal group and all fission neutrons pass through all the epithermal energy groups before becoming thermalized. A more general formulation is necessary for the treatment of fast reactors or reactors in which an important fraction of the fissions occur at epithermal energies. Applications and methods of solution of multigroup calculations are varied and are often too complicated to be handled by manual computation. In the following article the method is applied to the case of a bare reactor, which requires only a modest amount of computation but suffices to illustrate the application of the method.

10.1 The Multigroup Equations

If those neutrons whose energy lies between two arbitrary limits E_j and E_{j-1} ($E_{j-1} > E_j$) are considered to make up the flux in group j , the time-independent diffusion equation for this group can be written in the form

$$D_j \nabla^2 \phi_j(r) - (\Sigma_{c,j} + \Sigma_{f,j} + \Sigma_{in,j} + \Sigma_{el mod,j}) \phi_j(r) + S_j(r) = 0 \quad (214)$$

where $D_j \nabla^2 \phi_j(r)$ = usual leakage term

$\Sigma_{c,j}$ = macroscopic capture cross section for group j

$\Sigma_{f,j}$ = macroscopic fission cross section for group j

$\Sigma_{in,j}$ = macroscopic cross section for scattering inelastically out of group j

$\Sigma_{el mod,j}$ = macroscopic cross section for elastic moderation out of group j

$\Sigma_{remove,j} = \Sigma_{c,j} + \Sigma_{f,j} + \Sigma_{in,j} + \Sigma_{el mod,j}$ = the total macroscopic cross section for processes removing neutrons from group j

and where $S_j(r)$ includes all neutrons born into group j at position r . These neutrons could be the result of fission by neutrons of any and all energies or the result of an energy-degradation process from some higher energy into the band between E_j and E_{j-1} . (Herein the possibility of an external source will be omitted.)

The entire energy range of interest will be divided into N groups or energy bands. Furthermore, γ_j will be defined as the fraction of all fission neutrons born into group j , and ν_k as the total number of fission neutrons born per fission in any group k . Then Eq. (214) may be rewritten in the form

$$D_j \nabla^2 \phi_j(r) - \Sigma_{\text{remove},j} \phi_j(r) + \gamma_j \sum_{k=1}^N \nu_k \Sigma_{f,k} \phi_k(r) + \sum_{l=1}^{j-1} \Sigma_{in}(l \rightarrow j) \phi_l(r) + \sum_{m=1}^{j-1} \Sigma_{el\ mod}(m \rightarrow j) \phi_m(r) = 0 \quad (215)$$

Of course, $\Sigma_{in}(l \rightarrow j)$ is the macroscopic cross section for scattering from group l into group j . Thus

$$\sum_{j=1}^{l+1} \Sigma_{in}(l \rightarrow j) = \Sigma_{in,l} \quad (216)$$

and a similar expression holds for the elastic moderation cross sections. The elastic moderation source, in principle, could have contributions from all higher energy groups; hence it has been summed from the highest energy group 1 to that just above group j . However, for materials other than hydrogen, such contributions are possible only from relatively nearby energy bands and frequently only from the next higher (in energy) group.

Equation (215) then describes the diffusion of neutrons in group j within a particular homogeneous region. A set of cross sections for all N groups is required for this region. The result is N simultaneous equations which describe the problem over the entire range. Of course, if more than one region needs to be considered, a cross-sectional set is required for each region. The usual boundary conditions of continuity of flux and current are applied to each group, and the multiple sets of simultaneous equations solved.

This latter problem of multigroup theory in more than one region cannot generally be solved analytically. A numerical solution of an iterative type is required. This solution is of such magnitude and needs such great accuracy that only modern high-speed computing machinery is really practical for its achievement.

10.2 Simplifications for Single-region Computation

If multigroup diffusion theory is being applied to a single region (specifically, a bare reactor core) a simplifying assumption can be made which makes possible a semi-analytic solution with only a few hours work at a desk computer.

The basic hypothesis is that the flux has the same shape in every group. Within the framework of diffusion theory, this would be rigorously true, independent of geometry, if and only if the extrapolation distance were the same for all groups. This is not usually the case. Furthermore, there is direct experimental evidence that the neutron energy spectrum is not space independent in a bare core. Nevertheless, the assumption of a space-independent neutron energy spectrum in a bare core is not a very bad one and gives results which are rather good away from the boundaries.

Thus, it is assumed that

$$\begin{aligned} \phi_1(r) &= \phi_1 F(r) \\ \phi_k(r) &= \phi_k F(r) \\ \phi_l(r) &= \phi_l F(r) \end{aligned}$$

In other words, $F(r)$ is a spatial shape factor independent of energy. It is further

assumed that

$$\nabla^2 F(r) = -B^2 F(r) \quad (217)$$

where B^2 is the so-called material buckling. If a bare core of this material were built in any shape, its extrapolated dimensions would have to be such that the geometric buckling equaled this material buckling at criticality.

With these assumptions Eq. (215) becomes

$$-D_j B^2 \phi_j - \Sigma_{remove,j} \phi_j + \gamma_j \sum_{k=1}^N \nu_k \Sigma_{f,k} \phi_k + \sum_{l=1}^{j-1} \Sigma_{in}(l \rightarrow j) \phi_l + \sum_{m=1}^{j-1} \Sigma_{el\ mod}(m \rightarrow j) \phi_m = 0 \quad (218)$$

The spatial dependence has been completely removed, since it is now common to every term. It is convenient to rewrite Eq. (218) in another form, namely,

$$\phi_j = \frac{\gamma_j \sum_{k=1}^N \nu_k \Sigma_{f,k} \phi_k + \sum_{l=1}^{j-1} \Sigma_{in}(l \rightarrow j) \phi_l + \sum_{m=1}^{j-1} \Sigma_{el\ mod}(m \rightarrow j) \phi_m}{D_j B^2 + \Sigma_{remove,j}} \quad (219)$$

10.3 Procedure for Solution

Equation (219) really comprises a set of N simultaneous equations, one for each group. In practice, this is usually solved by an iterative procedure. For the highest energy group ($j = 1$) the numerator consists only of the fission source, no contributions being possible from inelastic or elastic moderation processes. If it is now

assumed that the total fission source for all groups $\sum_{k=1}^N \nu_k \Sigma_{f,k} \phi_k$ is some arbitrary number, say unity, ϕ_1 is given by

$$\phi_1 = \frac{\gamma_1}{D_1 B^2 + \Sigma_{remove,1}} \quad (220)$$

The quantity ϕ_1 still remains a function of B^2 . The procedure for solution is to guess some reasonable value for B^2 , keeping in mind that it is this quantity which will be iterated on. When the solution converges, the critical value of the material buckling will have been determined. The quantity ϕ_1 is computed from Eq. (220) with the trial value of B^2 ; it is then possible to evaluate ϕ_2 :

$$\phi_2 = \frac{\gamma_2 + \Sigma_{in}(1 \rightarrow 2) \phi_1 + \Sigma_{el\ mod}(1 \rightarrow 2) \phi_1}{D_2 B^2 + \Sigma_{remove,2}} \quad (221)$$

Since this group gets source neutrons other than fission only from group 1, it is completely specified and ϕ_2 can be calculated. The process is continued through all the groups in numerical order in this fashion.

At this point a set of ϕ_j 's has been obtained, and the convergence test is to be applied.

This is done by calculating a value for the total fission source $\sum_{k=1}^N \nu_k \Sigma_{f,k} \phi_k$ and comparing it with the original assignment of unity. If the two agree within the desired accuracy, the equations have been solved. Otherwise, a new guess on B^2 is made and a second iteration performed. The fission source is again computed, using the new set

of ϕ_i 's, and compared with unity. This process is repeated until convergence, which comes quickly with some practice.

When the solution is complete, not only the material buckling B^2 but also the neutron energy spectrum have been determined. The spectrum is, of course, given by the relative values of the total flux ϕ_i in each energy band.

10.4 Applications

The single-region calculation has a variety of applications, particularly for intermediate and fast reactors. It can be used in parameter studies to give the neutron energy spectra and critical masses for cores of varying composition (assuming a previous knowledge of reflector savings). In connection with the latter, some caution must be exercised in applying the results to reflected assemblies, since the reflector savings may differ sufficiently to alter the apparent trends of the one-region calculations.

The neutron energy spectrum can be used in predicting various performance criteria, also in establishing a set of one or two group cross sections for performing multiregion calculations. Caution must again be exercised in that this spectrum is quite accurate in performing energy averages for the core but will probably not be representative of the reflector spectrum. This latter will probably change quite rapidly with distance from the core.

10.5 Example

Suppose it is desired to determine the critical mass and various other performance data of a certain fast power reactor. It is sodium-cooled, uses a plate-type fuel element having a partly enriched uranium meat and a stainless-steel jacket. It is assumed that previous experience indicates a reasonable reflector saving for this core and the type of blanket contemplated. Then the material buckling is required for the assignment of critical mass.

First, a group energy structure must be chosen so that cross sections can be assigned. The choice is arbitrary, and the structure is a function of personal taste and previous experience. However, the technical facts do suggest that certain boundaries are reasonable. For example, the fission of U^{238} will be an important contributor to reactivity; also, it is an important performance criterion. Furthermore, U^{238} has a fission threshold in the neighborhood of 1 Mev; that is, it does not fission below this energy. Above, it rises rapidly to a rough plateau. Hence, a logical place for a group boundary is this threshold.

Again, suppose the fission or capture cross section of the fuel U^{235} exhibited a marked change below some energy. It might be quite flat above 100 keV but rise sharply below this energy. This, too, would then be a logical place for a group boundary. Indeed, if it was expected that a significant amount of the total flux present was to be found below 100 keV, several groups might be used below 100 keV in order to describe these events accurately. Thus, some idea of the neutron energy spectrum to be expected is helpful in assigning group structure.

Again, if one of the materials had a large resonance in a region of considerable flux, a group might be assigned to bracket it.

Finally, the manner in which certain important cross-sectional measurements were made might be suggestive of group structure. For example, to find where in the energy scale inelastically scattered neutrons go, threshold detectors like U^{238} and Np^{237} are used. The neptunium threshold is around 400 keV. If a significant amount of the inelastic scattering data is based on this detector, this energy becomes a natural position for a group boundary.

The above is only illustrative, not all-inclusive. Depending on the particular subject of interest, one or another group structure may be chosen.

In the present example, there is no special side interest which requires a bunching of groups, etc. A total of six groups is selected as a reasonable number to work with. Since most of the neutrons are expected in the neighborhood of 100 to 300 keV, a

greater density of groups is used in that region. Thus, the following group energy structure is chosen, and the γ_i 's are assigned accordingly.

Group No.	Energy interval, Mev	γ
1	$\infty - 1.4$	0.47
2	1.4-0.7	0.31
3	0.7-0.4	0.11
4	0.4-0.2	0.07
5	0.2-0.05	0.04
6	0.05-0.0	0

The γ_i 's are obtained by integrating the normalized fission spectrum with respect to energy between the limits of the various bands. Of course, $\sum_{j=1}^n \gamma_j = 1$.

One now needs to assign group cross sections. The technique used depends on the accuracy desired and the information available on spectrum and cross sections. One does not, a priori, know the shape of the flux in each group. Some assumption must be made, and average cross sections computed. In principle, if the cross sections are well known as a function of energy, one might get greater accuracy by using a large number of groups and iterating on the cross sections. That is, assume some flux shape in each group, find average cross sections, and solve the particular "bare core," getting a neutron energy spectrum. Now use this spectrum to reassign average cross sections in each group, and again solve the bare core problem.

In practice, cross sections may not be too well known, and a reasonable assumption on flux shape is sufficient. For a fast power reactor, a reasonable flux assumption for our particular group structure is that the flux is constant with energy in every group except the first while it resembles the fission spectrum in this energy region.

It is usually convenient to assign average group cross sections to each material, then combine them in the appropriate proportions. This technique is accurate for all cross sections except transport, where some difficulties arise if there are wide fluctuations within a group. However, for a mixture of elements, such fluctuations are usually minimized, so that the transport cross section can be given the same averaging process as the others.

Thus we may write the average fission cross section in group j for material x as

$$\sigma_{f,j}^x = \frac{\int_{E_{j-1}}^{E_j} \sigma_f^x(E) \phi(E) dE}{\int_{E_{j-1}}^{E_j} \phi(E) dE}$$

A similar expression applies for other cross sections of interest.

For inelastic scattering, it is frequently convenient to divide the description of this process into two parts. First, the average inelastic scattering out of each group is obtained. Then a matrix of coefficients is assigned to describe where in energy these neutrons go. We may write

$$\Sigma_{in,l}^x(l \rightarrow j) = \Sigma_{in,l}^x C_{l \rightarrow j}$$

where $\Sigma_{in,l}^x$ is the average macroscopic inelastic scattering cross section out of group l for material x and

$$\sum_{j=l+1}^n C_{l \rightarrow j} = 1$$

The summation is over all groups lower in energy than group l .

For the materials of the example and the particular group energy structure chosen, the elastic moderation from any group will contribute only to the group next lower in energy. The group cross section is, of course, a function of the average loss in the logarithm of the energy ξ , the scattering cross section Σ_{sc} , and the shape of the flux within the group. Again, it is possible to iterate on cross sections or to use approximations thereon. However, the previous assumption of a flat flux in every group but the first is quite reasonable for this example.

One possible method of procedure then is to compute a flux-weighted average energy \bar{E}_j for each group. One may then write for material x

$$\Sigma_{el\ mod.,j}^x = \frac{\xi^x \Sigma_{sc,j}^x}{\ln \bar{E}_j / \bar{E}_{j+1}}$$

A final note on cross sections is that the transport cross section used in the definition of the diffusion coefficient ($D = 1/3\Sigma_{tr}$) is the sum of the elastic transport $\Sigma_{sc}(1 - \mu_0)$ and the removal cross sections.

If one goes through the process of obtaining group cross sections and then averaging them appropriately in accordance with composition, one obtains a set of multigroup cross sections for the particular core under consideration. For purposes of illustration, a set of cross sections representative of a small, fast power reactor will be assumed and two iterations on buckling B^2 will be made.

No. group	Σ_{tr}	Σ_f	Σ_c	Σ_{rn}	$\Sigma_{el\ mod}$	Σ_{remove}
1	0.16	0.014	0.0010	0.050	0.012	0.077
2	0.19	0.010	0.0017	0.020	0.006	0.0377
3	0.27	0.010	0.0025	0.012	0.012	0.0365
4	0.28	0.011	0.0037	0.007	0.009	0.0307
5	0.35	0.016	0.0059	0.001	0.009	0.0319
6	0.41	0.026	0.0154	0.0414

The quantity ν_j will be taken as 2.5 for all groups. Furthermore, for simplicity, it will be assumed that the matrix of coefficients which prescribes the destination of inelastically scattered neutrons is the same for all materials. We take these to be as follows:

From \ To	1	2	3	4	5
2	0.4				
3	0.3	0.65			
4	0.15	0.2	0.7		
5	0.1	0.1	0.2	0.8	
6	0.05	0.05	0.1	0.2	1

As previously mentioned, the elastic moderation for this example is always into the next lower energy group, so that no corresponding matrix is required.

One makes a first guess on B^2 usually from previous experience. If no such background exists, a rough one-group set of cross sections, averaged over composition, will give a reasonable first guess, using the formula

$$B^2 = \frac{\nu \Sigma_f - \Sigma_c}{D}$$

In the present example, $B^2 = 0.01$ will be tried for the first iteration. As is shown in Table 15, the fission source calculated using the first set of ϕ_j 's is 1.03414. Hence, the result is 3.414 per cent away from convergence. A second guess of $B^2 = 0.0105$ is made, and this time the result is only 0.95 per cent away from convergence. If this were not satisfactory, a third iteration would be made.

Table 15. Bare Core Multigroup Calculations

j	γ_j	$\Sigma_{in}(1 \rightarrow j)\phi_1$	$\Sigma_{in}(2 \rightarrow j)\phi_2$	$\Sigma_{in}(3 \rightarrow j)\phi_3$	$\Sigma_{in}(4 \rightarrow j)\phi_4$	$\Sigma_{in}(5 \rightarrow j)\phi_5$	$\gamma_j + \sum_{i=1}^{j-1} \Sigma_{in}(i \rightarrow j)\phi_i$	$\Sigma_{el\ mod.\ j-1}$	$\phi_{j-1}\Sigma_{el\ mod.\ j-1}$	Col. 7 + col. 9	$3\Sigma_{tr,j}$	$\Sigma_{a,j}$	$\frac{B^2}{3\Sigma_{tr,j}} + \Sigma_{a,j}$	$\phi_j = \frac{\text{col. 10}}{\text{col. 13}}$
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)
1	0.47	0.47	0.47	0.48	0.077	0.097833	4.8041
2	0.31	0.096082	0.406082	0.012	0.057649	0.463731	0.57	0.0377	0.055244	8.3942
3	0.11	0.072062	0.109125	0.291187	0.006	0.050365	0.341552	0.81	0.0365	0.048846	6.9924
4	0.07	0.036031	0.033577	0.058736	0.198344	0.012	0.083909	0.282253	0.84	0.0307	0.042605	6.6249
5	0.04	0.024020	0.016788	0.016782	0.037099	0.134689	0.009	0.059624	0.194313	1.05	0.0319	0.041424	4.6908
6	0	0.012010	0.008394	0.008391	0.009275	0.004691	0.042761	0.009	0.042217	0.084978	1.23	0.0414	0.049530	1.7156
$\sum_{j=1}^6 \nu_j \Sigma_{f,j} \phi_j = 1.03414$														
Second iteration. Try $B^2 = 0.0105$														
1	0.47	0.47	0.47	0.098875	4.7535
2	0.31	0.95070	0.405070	0.012	0.057042	0.462112	0.056121	8.2342
3	0.11	0.071302	0.107045	0.288347	0.006	0.049405	0.337752	0.049463	6.8284
4	0.07	0.035651	0.032937	0.057359	0.195947	0.012	0.081941	0.277888	0.043200	6.4326
5	0.04	0.023768	0.016468	0.016388	0.036023	0.132647	0.009	0.057893	0.190540	0.041900	4.5475
6	0	0.011884	0.008234	0.008194	0.009006	0.004548	0.041866	0.009	0.040928	0.082794	0.049937	1.6580
$\sum_{j=1}^6 \nu_j \Sigma_{f,j} \phi_j = 1.0095$														

11 SAMPLE CALCULATIONS

The following calculations illustrate the procedures involved in calculating the material constants and reactivity for a typical lumped reactor. Since the purpose is to illustrate procedures rather than to arrive at very accurate results, the numbers involved are carried out to slide-rule accuracy. In making practical calculations it will usually be found desirable to use a desk-type calculating machine and to carry more significant figures, particularly in calculations of the type covered in Arts. 11.5 and 11.6.

Again, to illustrate procedures, values of some of the reactor constants are calculated even though applicable measured values exist. It should be remembered that the methods used here are only approximate, and when great accuracy is required, measured values must be sought or higher order approximations to transport theory must be used.

11.1 Description of the Reactor

The sample reactor is taken to be one moderated by D_2O , using as fuel natural uranium in the form of circular rods 2 cm in diameter, cased in aluminum of 0.05-cm thickness. The rods are arranged in a triangular lattice, with uniform spacing of 16 cm (Figs. 16 and 17). The core of the reactor is a circular cylinder, 200 cm in diameter and 200 cm high, and is surrounded on all sides by a reflector of pure D_2O , 30 cm thick (Figs. 18 and 19). The reactor is assumed to be at room temperature.

The area of the triangle drawn in Fig. 17 is 110.9 cm². The moderator contained in a tri-

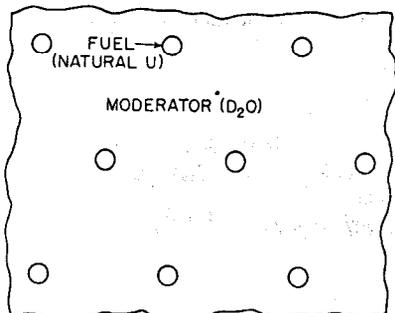


FIG. 16. Section of reactor for sample calculations.

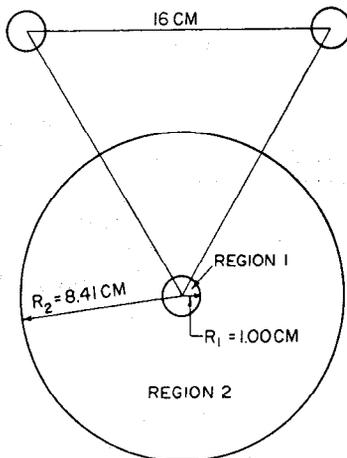


FIG. 17. Cell arrangement for sample calculations.

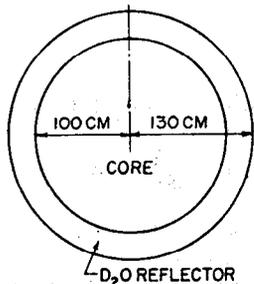


FIG. 18. Horizontal cross section of sample reactor.

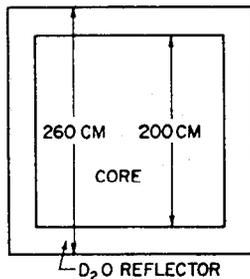


FIG. 19. Vertical cross section of sample reactor.

angular prism of this cross section and unit height is associated with the volume of uranium contained in half a fuel-rod section of unit length (1.571 cm³) and with the volume of aluminum making up half a can of unit length (0.16 cm³). From these numbers the volume ratio of moderator to fuel and the radius of the equivalent circular cell R_2 can be computed. These quantities, the atomic density in the uranium region N_1 and in the aluminum region N_3 , and the molecular density in the moderator region N_2 are tabulated below. The density of uranium is taken as 18.5 g/cm³ and that of D₂O as 1.10 g/cm³.

$$\frac{\text{Vol. D}_2\text{O}}{\text{Vol. U}} = \frac{110.9 - 1.571 - 0.16\dagger}{1.571} = 69.4$$

$$\frac{\text{Vol. Al}}{\text{Vol. U}} = \frac{(\pi)(2)(0.05)}{\pi(1)^2} = 0.10$$

$$R_1 = 1.00 \text{ cm}$$

$$R_2 = \sqrt{\frac{(2)(110.9)}{\pi}} = 8.41 \text{ cm}$$

$$N_1 = \frac{(18.5)(0.603 \times 10^{24})}{238} = 0.0469 \times 10^{24} \text{ atoms/cm}^3$$

$$N_2 = \frac{(1.10)(0.603 \times 10^{24})}{18.0} = 0.0368 \times 10^{24} \text{ molecules/cm}^3$$

$$N_3 = \frac{(2.70)(0.603 \times 10^{24})}{27.0} = 0.0603 \times 10^{24} \text{ atoms/cm}^3$$

11.2 Infinite Multiplication Constant k

11.21 Regeneration Factor η . † Designate U²³⁵ by the number 25 and U²³⁸ by the number 28; then

$$\begin{aligned} \eta &= \frac{\nu(25)\Sigma_f(25)}{\Sigma_a(25) + \Sigma_a(28)} \quad \text{by Eq. (169)} \\ &= \frac{\nu(25)N_{25}\sigma_f(25)}{N_{25}\sigma_a(25) + N_{28}\sigma_a(28)} = \frac{\nu(25)\sigma_f(25)}{\sigma_a(25) + (N_{28}/N_{25})\sigma_a(28)} \\ &= \frac{(2.46)(0.981)(580)}{(0.981)(687) + (0.993/0.00715)(2.75)} = 1.326 \end{aligned}$$

11.22 Thermal Utilization f :

$$\begin{aligned} \Sigma_{a1} &= N_1[0.00715\sigma_a(25) + 0.993\sigma_a(28)] \\ &= 0.0469 \times 10^{24}[(0.00715)(0.981)(687)(10^{-24}) + (0.993)(2.75)(10^{-24})] \frac{1}{1.128} \\ &= 0.314 \end{aligned}$$

[See Eq. (65).]

$$\Sigma_{a2} = \frac{\lambda_{tr}}{3L^2(D_2O)} = \frac{2.65}{(3)(116)^2} = 6.57 \times 10^{-6} \text{ cm}^{-1} \quad \text{from Table 13}$$

$$\Sigma_{a3} = N_3\sigma_a(\text{Al}) = (0.0603)(10^{24})(0.230)(10^{-24}) \left(\frac{1}{1.128} \right) = 0.0123 \text{ cm}^{-1}$$

† An inconsistent number of significant figures is carried in the equations to aid the reader in recognizing the numbers.

‡ All cross sections are from BNL-325. Since only ratios of cross sections are involved in computing η , the 2,200-m/sec cross sections can be used instead of values averaged over the Maxwell distribution. However, it is necessary to include the correction (0.981) for the departure of the U²³⁵ cross sections from a $1/v$ variation (see BNL-325).

$$\kappa_1 = \frac{1}{L_1} = \frac{1}{1.55} \frac{18.5}{18.9} = 0.632$$

$$\kappa_2 = \frac{1}{L_2} = \frac{1}{116} = 0.00862 \quad \text{from Table 13}$$

$$\kappa_1 R_1 = (0.632)(1) = 0.632$$

$$\kappa_2 R_2 = (0.00862)(8.41) = 0.0725$$

$$\kappa_2 R_1 = (0.00862)(1) = 0.00862 \quad \frac{\kappa_2(R_2^2 - R_1^2)}{R_1} = \frac{(0.00862)(8.41)^2 - (1)^2}{(1)^2} = 0.601$$

$$E = \frac{\kappa_2(R_2^2 - R_1^2)}{2R_1} \frac{I_0(\kappa_2 R_1)K_1(\kappa_2 R_2) + K_0(\kappa_2 R_1)I_1(\kappa_2 R_2)}{I_1(\kappa_2 R_2)K_1(\kappa_2 R_1) - K_1(\kappa_2 R_2)I_1(\kappa_2 R_1)} \quad \text{from Table 8}$$

$$= \frac{0.601}{2} \frac{I_0(0.00862)K_1(0.0725) + K_0(0.00862)I_1(0.0725)}{I_1(0.0725)K_1(0.00862) - K_1(0.0725)I_1(0.00862)}$$

$$= 0.301 \frac{(1.000)(13.70) + (4.87)(0.0363)}{(0.0363)(116) - (13.70)(0.00431)} = 1.0067$$

$$F = \frac{\kappa_1 R_1}{2} \frac{I_0(\kappa_1 R_1)}{I_1(\kappa_1 R_1)} = \frac{0.632}{2} \frac{I_0(0.632)}{I_1(0.632)} = 0.316 \frac{1.1024}{0.332} = 1.049$$

$$\frac{1}{f} = 1 + \frac{V_2 \Sigma_{a2} + V_3 \Sigma_{a3}}{V_1 \Sigma_{a1}} F + (E - 1) \quad \text{from Eq. (190)}$$

$$\frac{1}{f} = 1 + \left[\frac{(69.4)(6.57 \times 10^{-5}) + (0.10)(0.0123)}{(1)(0.314)} \right] (1.049) + 0.0067 = 1.0261$$

$$f = 0.9746$$

11.23 Resonance Escape Probability (p):

$$\kappa_1 = (0.0222)(18.5) = 0.411 \text{ cm}^{-1} \quad \text{from Table 10}$$

$$\kappa_2 = (0.141)(1.10) = 0.155 \text{ cm}^{-1} \quad \text{from Table 11}$$

$$\kappa_1 R_1 = 0.411 \quad \kappa_2 R_2 = 1.304 \quad \kappa_2 R_1 = 0.155$$

$$E = \frac{0.155}{2} \frac{(70.7 - 1)}{1} \frac{I_0(0.155)K_1(1.304) + K_0(0.155)I_1(1.304)}{I_1(1.304)K_1(0.155) - K_1(1.304)I_1(0.155)}$$

$$= 5.40 \frac{(1.006)(0.370) + (2.00)(0.801)}{(0.801)(6.26) - (0.3703)(0.0777)} = 2.13$$

$$F = \frac{0.411}{2} \frac{I_0(0.411)}{I_1(0.411)} = \frac{0.411}{2} \frac{1.043}{0.210} = 1.020$$

$$\frac{S}{M} = \frac{\text{surface area of } U}{\text{mass of } U} = \frac{2\pi R_1}{\pi R_1^2 \rho} = \frac{2}{R_1 \rho} = \frac{2}{(1)(18.5)} = 0.108 \text{ cm}^2/\text{g}$$

$$\int \sigma_{a,ff}(E') \frac{dE'}{E'} = A + \mu \frac{S}{M} = [9.25 + (24.7)(0.108)](10^{-24})$$

$$= 11.92 \times 10^{-24} \text{ cm}^2/\text{U}^{238} \text{ atom} \quad \text{by Table 9}$$

$$N_{28} = N_1(0.993) = (0.0469 \times 10^{24})(0.993) = 0.0466 \times 10^{24} \text{ atoms/cm}^3$$

$$\bar{\xi} \Sigma_s = 5.28 \times 10^{-24} \text{ cm}^2/\text{D}_2\text{O molecule} \quad \text{by Table 11}$$

$$\bar{\xi} \Sigma_s = (5.28 \times 10^{-24})(0.0368 \times 10^{24}) = 0.194 \text{ cm}^{-1}$$

$$p = \exp \left[- \frac{1}{\frac{V_2}{V_1} \frac{\bar{\xi} \Sigma_s F}{N_{28} \int \sigma_{a,ff}(E') (dE'/E')} + (E - 1)} \right] \quad \text{by Eq. (206)}$$

$$= \exp \left[- \frac{1}{\frac{(69.4)(0.194)(1.020)}{(0.0466 \times 10^{24})(11.92 \times 10^{-24})} + 2.13 - 1} \right] = e^{-0.0387} = 0.962$$

11.24 Fast Fission Factor ϵ :

$$\kappa_1(\text{thermal}) = \frac{1}{L} = \frac{1}{1.55} \left(\frac{18.5}{18.9} \right) = 0.632 \quad \text{from Table 13}$$

$$\Sigma_t = (0.0466)(4.3) = 0.200$$

$$\Sigma_f = (0.0466)(0.29) = 0.0135$$

$$\Sigma_{st} = (0.0466)(1.5) = 0.070$$

$$\Sigma_c = 0$$

$$\nu = 2.55$$

$$\frac{\kappa_1}{\Sigma_t} = \frac{0.632}{0.200} = 3.16$$

$$P' = 0.21 \quad P = 0.21 \quad \text{from Fig. 15}$$

$$\epsilon - 1 = \frac{[\nu - 1 - (\Sigma_c/\Sigma_f)](\Sigma_f/\Sigma_t)P}{1 - \frac{(\nu\Sigma_f + \Sigma_{st})}{\Sigma_t}P'} \quad \text{by Eq. (207)}$$

$$= \frac{(2.55 - 1)(0.0135/0.200)(0.21)}{1 - \left[\frac{(2.55)(0.0135) + 0.070}{0.200} \right] (0.21)} = 0.025$$

11.25 Infinite Multiplication Constant k :

$$k = \eta\epsilon pf = (1.326)(1.025)(0.962)(0.975) = 1.275$$

11.3 The Diffusion Coefficients D_f and D_s

Since the volume of metal in the reactor core is less than 2 per cent of the total volume, the effect of the metal on the diffusion coefficients is neglected, and D_s for both core and reflector is taken from Table 13:

$$D_s = \frac{\lambda_{sr}}{3} = \frac{2.65}{3} = 0.883$$

D_f may be computed by Eq. (213). It is evident, however, from the cross section curves for D and O, that the average values for the scattering cross sections between, say, 2 Mev and thermal energy are very nearly

$$\bar{\sigma}_s(D) = 3.4 \times 10^{-24} \quad \bar{\sigma}_s(O) = 3.6 \times 10^{-24}$$

using these average values:

$$D_f = \frac{\lambda_{sr}}{3} = \frac{1}{3} \left\{ \frac{1}{[1 - \bar{\mu}_0(D)]\Sigma_s(D) + [1 - \bar{\mu}_0(O)]\Sigma_s(O)} \right\}$$

where

$$\bar{\mu}_0 = \frac{2}{3A} \quad \text{by Eq. (11)}$$

$$1 - \bar{\mu}_0(D) = 0.666 \quad 1 - \bar{\mu}_0(O) = 0.958$$

$$D_f = \frac{1}{3} \left[\frac{1}{(0.666)(0.0368 \times 10^{24})(2)(3.4 \times 10^{-24}) + (0.958)(0.0368 \times 10^{24})(3.6 \times 10^{-24})} \right] \\ = 1.14 \text{ cm}$$

11.31 The Diffusion Area L^2 and Age τ . By Eq. (211)

$$L^2 = L_2^2(1 - f) = (116)^2(0.0254) = 342 \text{ cm}^2$$

The age τ for the metal-D₂O mixture can be taken to be the same as that for D₂O.

From Table 14,

$$\tau = 125 \text{ cm}$$

11.4 The Material Buckling B_m^2

The material buckling is given by the modified one-group equation as

$$B_m^2 = \frac{k - 1}{M^2} = \frac{0.275}{467} = 5.89 \times 10^{-4} \text{ cm}^{-2}$$

This is a relatively large value for which the modified one-group equation can be expected to give only a fair approximation. The characteristic equation based on the Fermi age equation is a much closer approximation:

$$(1 + L^2 B^2) e^{\tau B^2} = k$$

This equation can be solved by trial and error or otherwise to give

$$B_m^2 = 5.50 \times 10^{-4} \text{ cm}^{-2}$$

11.5 Critical Size of Bare Reactor

This quantity is not required for the solution of the problem under consideration, but the computation is included for the sake of completeness. If the reactor is assumed cylindrical in shape, the ratio of length H to radius R must be specified arbitrarily. Assume here that it is 2.

$$B_g^2 = \frac{\pi^2}{H^2} + \frac{(2.405)^2}{R^2} = \frac{\pi^2}{(2R)^2} + \frac{(2.405)^2}{R^2} \quad \text{from Table 5}$$

$$B_g^2 = B_m^2 = 5.50 \times 10^{-4} \text{ cm}^{-2} \quad \text{by the criticality condition}$$

$$R^2 = \frac{1}{5.50 \times 10^{-4}} \left[\frac{\pi^2}{4} + (2.405)^2 \right] = 1.500 \times 10^4$$

$$R = 122.5 \text{ cm} \quad H = 245 \text{ cm}$$

11.6 Reactivity of Reflected Reactor

In the following, the method of Art. 7.1 will be employed. Since the two-group formulation is less accurate than that of the Fermi age for D_2O -moderated reactors, the first step will be to determine an effective slowing-down area L_f which will make the two-group material buckling equal to the Fermi age material buckling, i.e.,

$$(1 + L^2 B^2)(1 + L_f^2 B^2) = (1 + L^2 B^2) e^{\tau B^2}$$

$$L_f^2 = \frac{e^{\tau B^2} - 1}{B^2} = \frac{0.0735}{5.50 \times 10^{-4}} = 134.5 \text{ cm}^2$$

Since the problem must be reduced to a one-dimensional problem, the choice is arbitrarily made to examine first the radial solution, using an estimated equivalent bare reactor solution in the axial direction. A guess is made for the axial reflector saving. Because D_2O is a good reflector, a reasonable guess for the reflector saving on each end of the core would be just slightly less than the actual reflector thickness—say 25 cm. The first estimate (which will be reexamined later) for the partial buckling in the axial direction is, then,

$$\text{Axial partial buckling} = \frac{\pi^2}{[200 + 2(25)]^2} = 1.580 \times 10^{-4} \text{ cm}^{-2}$$

The remainder of the problem consists of adjusting one or more of the reactor constants to a value which will make the critical determinant [Eq. (144)] vanish. In the following, k , the infinite multiplication constant, will be varied arbitrarily to achieve this condition. Although such a variation of k alone is not a physically possible procedure (e.g., if k were changed by changing fuel enrichment, L^2 would also change), the determination of the critical value of k is a useful operation, since a comparison of k (critical) with k (actual) yields a determination of the reactivity of the system. To proceed with the calculation, a trial value of k must be arbitrarily chosen. Since D_2O is known to be an effective reflector and the reflector is relatively thin, a reasonable first choice is that value of k which would make the reactor critical if the reflector saving were equal to the reflector thickness. This value of k should be near the true critical value, but almost certainly a little too small. Thus the direction to go in choosing a second trial value of k will be known. The specified first trial value is $k = 1.245$.

In Table 16 are listed constants which will be used in the calculation; these have been evaluated in the preceding sections or are derived from constants evaluated there. The notation is that of Art. 7. The slight variations from previous notation are necessary to avoid an excessive number of subscripts in writing the complex equations involved.

Table 16

Core constants (region 1)	Reflector constants (region 2)
$k = 1.245$ (trial value)	
$R_1 = 100$ cm	$R_2 = 130$ cm (Fig. 5)
$L_{1a} = 342$ cm ²	$L_{2a}^2 = (116)^2 = 13,450$ cm ² (Table 13)
$L_{1f} = 134.5$ cm ²	$L_{2f}^2 = 134.5$ cm ²
$D_{1a} = 0.883$ cm	$D_{2a} = 0.883$ cm
$D_{1f} = 1.14$ cm	$D_{2f} = 1.14$ cm
$\Sigma_{1a} = D_1/L_{1a}^2 = 0.00258$ cm ⁻¹	$\Sigma_{2a} = 0.656 \times 10^{-4}$ cm ⁻¹
$\Sigma_{1f} = D_1/L_{1f}^2 = 0.00848$ cm ⁻¹	$\Sigma_{2f} = 0.00848$ cm ⁻¹
$\kappa_{1a}^2 = 1/L_{1a}^2 = 0.00292$ cm ⁻²	$\kappa_{2a}^2 = 0.744 \times 10^{-4}$ cm ⁻²
$\kappa_{1f}^2 = 1/L_{1f}^2 = 0.00744$ cm ⁻²	$\kappa_{2f}^2 = 0.00744$ cm ⁻²
$B_1^2 = 4.88 \times 10^{-4}$ cm ⁻²	$B_2^2 = -\kappa_{2a}^2 = -0.744 \times 10^{-4}$ cm ⁻²
[Eq. (109)]	
$B_1'^2 = -0.01085$ cm ⁻²	$B_2'^2 = -\kappa_{2f}^2 = -0.00848$ cm ⁻²
[Eq. (110)]	
$l^2 = B_1^2 - \left(\frac{\pi}{H + 2RS} \right)^2 = B_1^2 - 1.580 \times 10^{-4}$	$-\mu_a^2 = B_2^2 - 1.580 \times 10^{-4} = -2.324 \times 10^{-4}$
$= 3.30 \times 10^{-4}$	
$-m^2 = B_1'^2 - 1.580 \times 10^{-4} = 0.01101$	$-\mu_f^2 = B_2'^2 - 1.580 \times 10^{-4} = -0.00864$
$S_1 = \frac{\Sigma_{1a} + D_{1a}B_1^2}{\Sigma_{1f}} = 0.355$	
$S_1' = \frac{\Sigma_{1a} + D_{1a}B_1'^2}{\Sigma_{1f}} = 0.826$	$S_2' = \frac{\Sigma_{2a} - D_{2a}\kappa_{2f}^2}{\Sigma_{2f}} = -0.767$
$lR_1 = 1.818$	$\mu_a R_1 = 1.525$
$mR_1 = 10.50$	$\mu_f R_1 = 9.30$
	$\mu_a R_2 = 1.983$
	$\mu_f R_2 = 12.10$

The following Bessel functions will be needed to evaluate the critical determinant:

$$\begin{array}{ll}
 J_0(lR_1) = J_0(1.818) = 0.3295 & J_1(lR_1) = 0.5818 \\
 I_0(mR_1) = I_0(10.50) = 4,500 & I_1(mR_1) = 4,280 \\
 I_0(\mu_a R_1) = I_0(1.525) = 1.672 & I_1(\mu_a R_1) = 1.0067 \\
 I_0(\mu_a R_2) = I_0(1.983) = 2.253 & \\
 I_0(\mu_f R_1) = I_0(9.30) = 1,447 & I_1(\mu_f R_1) = 1,366 \\
 I_0(\mu_f R_2) = I_0(12.10) = 2.078 \times 10^4 & \\
 K_0(\mu_a R_1) = K_0(1.525) = 0.2070 & K_1(\mu_a R_1) = 0.2676 \\
 K_0(\mu_a R_2) = K_0(1.983) = 0.1163 & \\
 K_0(\mu_f R_1) = K_0(9.30) = 0.372 \times 10^{-4} & K_1(\mu_f R_1) = 0.392 \times 10^{-4} \\
 K_0(\mu_f R_2) = K_0(12.10) = 1.990 \times 10^{-5} &
 \end{array}$$

The critical determinant D is [Eq. (144)]

$$D = \begin{vmatrix} J_0(lR_1) & I_0(mR_1) & \frac{K_0(\mu_s R_2)}{I_0(\mu_s R_2)} I_0(\mu_s R_2) - K_0(\mu_s R_1) & \frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_0(\mu_f R_2) - K_0(\mu_f R_1) \\ S_1 J_0(lR_1) & S_1' I_0(mR_1) & 0 & S_2' \left[\frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_0(\mu_f R_2) - K_0(\mu_f R_1) \right] \\ D_{1s} J_1(lR_1) & -D_{1s} m I_1(mR_1) & -D_{2s} \mu_s \left[\frac{K_0(\mu_s R_2)}{I_0(\mu_s R_2)} I_1(\mu_s R_1) + K_1(\mu_s R_1) \right] & -D_{2s} \mu_f \left[\frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_1(\mu_f R_1) + K_1(\mu_f R_1) \right] \\ D_{1f} S_{1l} J_1(lR) & -D_{1f} S_{1l}' m I_1(mR_1) & 0 & -S_2' D_{2f} \mu_f \left[\frac{K_0(\mu_f R_2)}{I_0(\mu_f R_2)} I_1(\mu_f R_1) + K_1(\mu_f R_1) \right] \end{vmatrix}$$

Putting in the numerical values yields

$$D = \begin{vmatrix} 0.3295 & 4,500 & -0.1207 & -0.371 \times 10^{-4} \\ 0.1170 & -3,715 & 0 & 0.285 \times 10^{-4} \\ 0.00934 & -397 & -0.004304 & -0.0322 \times 10^{-4} \\ 0.00428 & 423 & 0 & 0.0319 \times 10^{-4} \end{vmatrix}$$

The determinant may be evaluated by expanding by minors. The result is

$$D = 0.0825 \times 10^{-4}$$

Since D is not zero, the assumed value of k is not the critical value. A larger value is chosen for the second trial. This is arbitrarily taken to be the actual k of the reactor core material ($k = 1.275$). Note that only the elements of the first two columns of the determinant must be recomputed. The new value of the determinant is $D' = -0.095 \times 10^{-4}$. Linear interpolation between D and D' yields, for the value of k which will make the determinant zero, $k(0) = 1.259$. If an extremely accurate value of $k(0)$ is required, a third trial computation should be made, using the value 1.259 for the trial value of k . The third trial will not be made here.

The value of $k(0)$ is not an accurate value for the critical k of the actual reactor but is the critical value for the reactor which has the assumed equivalent bare dimension in the axial direction [H (equiv) = 250 cm]. The useful quantity which can be evaluated with some precision is the radial component of the buckling. The critical material buckling of the assumed reactor is given by

$$B^2 = \frac{-(x_{1f}^2 + x_{1s}^2) + \sqrt{(x_{1f}^2 + x_{1s}^2)^2 + 4x_{1f}^2x_{1s}^2[k(0) - 1]}}{2}$$

$$= 5.20 \times 10^{-4} \text{ cm}^2$$

The axial buckling of the assumed reactor $(\pi/250)^2$ is $1.58 \times 10^{-4} \text{ cm}^2$; hence the radial buckling of the *critical* reactor must be

$$\text{Radial buckling} = 5.20 \times 10^{-4} - 1.58 \times 10^{-4} = 3.62 \times 10^{-4}$$

and the equivalent bare radius of the reflected reactor is

$$R \text{ (equiv)} = \frac{2.405}{\sqrt{3.62 \times 10^{-4}}} = 126 \text{ cm}$$

To obtain an accurate specification of the criticality condition, one must solve the problem of the cylindrical reactor, bare radially and reflected on the ends (Art. 7.2), using the above value for the equivalent bare radius. If reasonable guesses were made originally for the axial reflector saving, the two computations (one radial, one axial) should suffice to determine the critical value of k . If a very poor initial guess was made, a second radial computation may be necessary, since the two components of the buckling are not completely independent.

A reasonably accurate value for the critical k can be obtained without resorting to the axial calculation by assuming that the axial reflector saving (on each end of the core) is equal to the computed radial reflector saving. This is a rather good assumption if the reflector saving is considerably less than the core dimensions (provided, of course, the materials of axial and radial reflector are identical). The computed radial saving is

$$RS = R \text{ (equiv)} - R \text{ (actual)} = 126 - 100 = 26 \text{ cm}$$

The equivalent bare height of the reactor is, then,

$$H \text{ (equiv)} = 200 + 2(26) = 252 \text{ cm}$$

The corresponding buckling is

$$B^2 = 3.62 \times 10^{-4} + \left(\frac{\pi}{252}\right)^2 = 5.17 \times 10^{-4} \text{ cm}^2$$

and the critical value of k is given by

$$k(\text{crit}) = \left(1 + \frac{B^2}{\kappa_{1s}^2}\right) \left(1 + \frac{B^2}{\kappa_{1f}^2}\right) = 1.259$$

Because the original guess for the axial reflector saving was quite close, the value of $k(\text{crit})$ agrees with that of $k(0)$ to within the accuracy of the computation. Obviously, a change of a centimeter or two in reflector saving has no very great effect on the reactivity of a large reactor.

The original problem undertaken was to calculate the reactivity of the specified reactor. This is most conveniently expressed in terms of the excess reactivity $\delta k_{eff}/k_{eff}$, which is given by Eq. (261):

$$\frac{\delta k_{eff}}{k_{eff}} = \frac{\delta k}{k} = \frac{k(\text{actual}) - k(\text{crit})}{k(\text{actual})} = \frac{1.275 - 1.259}{1.275} = 0.0125$$

If it is necessary to make the reactor exactly critical, one or more of its characteristic properties (e.g., size, enrichment, lattice spacing) must be altered. In making such adjustments, if they do not change k_{eff} by more than 2 or 3 per cent, and if they do not alter the moderator density, it is usually safe to assume that they will not affect reflector savings appreciably.

11.7 Solution by Matrix Method

To illustrate the use of the matrix method of solution outlined in Sec. 8, the problem of the preceding section will be re-solved by that method. The same first guess will be made for $k(1.245)$, and all the constants listed in Table 16 will be applicable.

In order to construct the Q matrix for the reflector, the following elements are evaluated, using the constants of Table 16 in the expressions listed in Art. 8.2:

$$\frac{1}{D_{2f}} q(\mu_f) = \frac{1}{\mu_f} \frac{K_0(\mu_f R_1)}{K_1(\mu_f R_1)} \frac{1 - \frac{K_0(\mu_f R_2) I_0(\mu_f R_1)}{K_0(\mu_f R_1) I_0(\mu_f R_2)}}{1 + \frac{K_0(\mu_f R_2) I_1(\mu_f R_1)}{K_1(\mu_f R_1) I_0(\mu_f R_2)}} = 8.79$$

$$\frac{1}{D_{2s}} q(\mu_s) = \frac{1}{\mu_s} \frac{K_0(\mu_s R_1)}{K_1(\mu_s R_1)} \frac{1 - \frac{K_0(\mu_s R_2) I_0(\mu_s R_1)}{K_0(\mu_s R_1) I_0(\mu_s R_2)}}{1 + \frac{K_0(\mu_s R_2) I_1(\mu_s R_1)}{K_1(\mu_s R_1) I_0(\mu_s R_2)}} = 28.06$$

$$q(\mu_f, \mu_s) = \frac{1}{S_2' D_{2f}} q(\mu_f) - q(\mu_s) = 16.87$$

The Q matrix is (Art. 8.2):

$$Q = \begin{vmatrix} 1 & 0 & 8.79 & 0 \\ 0 & 1 & 16.87 & 28.06 \end{vmatrix}$$

Since there are only two regions involved in the problem, the Y_n matrices for intermediate regions are not required. The following elements are needed for the core matrix Y_{core} (Art. 8.4):

$$\frac{1}{S_1} = \frac{1}{0.355} = 2.818 \quad \frac{1}{S_1'} = \frac{1}{-0.826} = -1.210$$

$$D_{1f} X_c(m) = D_{1f} m \frac{I_1(mR_1)}{I_0(mR_1)} = 0.1138$$

$$\frac{D_{1s}}{S_1'} X_c(m) = \frac{D_{1s} m}{S} \frac{I_1(mR_1)}{I_0(mR_1)} = -0.1067$$

$$D_{1f} X_c(l) = 1.14 X_c(l) = 1.14 l \frac{J_1(lR_1)}{J_0(lR_1)}$$

$$\frac{D_{1s}}{S_1} X_c(l) = 2.49 X_c(l)$$

Note that although quantities are available in Table 16 for evaluating $X_c(l)$, we choose to retain it as a variable for the time being. The core matrix is, then,

$$Y_{\text{core}} = \begin{vmatrix} 1 & 1 \\ 2.818 & -1.210 \\ -1.14X_c(l) & 0.1138 \\ -2.49X_c(l) & -0.1067 \end{vmatrix}$$

The product QY_{core} is formed,† giving the matrix

$$QY_{\text{core}} = \begin{vmatrix} 1 - 10.14X_c(l) & 2.013 \\ 2.818 - 89.0X_c(l) & -2.300 \end{vmatrix}$$

The condition for criticality is that the determinant of the QY_{core} matrix be zero; i.e.,

$$\begin{vmatrix} 1 - 10.14X_c(l) & 2.013 \\ 2.818 - 89.0X_c(l) & -2.300 \end{vmatrix} = 0$$

The course chosen here for determining criticality is arbitrarily to set the determinant equal to zero and solve for $X_c(l)$. This specifies a value of l which will make the determinant zero. If the value so determined agrees with the trial value (Table 16), the condition of criticality has been determined. If it does not agree with the trial value, criticality has not been found, since no value of l other than the trial value is consistent with the other quantities used in the Y_{core} matrix.

Expanding the critical determinant yields

$$X_c(l) = 0.0394 = \frac{J_1(lR_1)}{J_0(lR_1)}$$

The above equation can be solved quickly by trial and error, giving

$$l = 0.01902$$

which differs from the trial value. The easiest procedure now is to use this value (0.01902) as a new trial value, recompute all the elements of the Y_{core} matrix, take the product QY_{core} , and test again for criticality. To determine the new elements consistent with the new trial value, recall that

$$B_1^2 = l^2 + \left(\frac{\pi}{H + 2RS} \right)^2 = l^2 + 1.580 \times 10^{-4}$$

$$B_1'^2 = -(B_1^2 + \kappa_1'^2 + \kappa_{1s}^2) \quad \text{from Eqs. (109) and (110)}$$

$$m^2 = -B_1'^2 + 1.580 \times 10^{-4}$$

etc.

The new core matrix is

$$Y_{\text{core}} = \begin{vmatrix} 1 & 1 \\ 2.79 & -1.206 \\ -1.14X_c(l) & 0.1138 \\ -2.45X_c(l) & 0.1064 \end{vmatrix}$$

Formation of the product QY_{core} and determination of the value of $X_c(l)$ corresponding to a zero value of the determinant give

$$X_c(l) = 0.0392 \quad l = 0.01900 \quad l^2 = 3.61 \times 10^{-4}$$

† The product of two matrices, for example,

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \text{and} \quad b = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix}$$

is defined as

$$ab = \begin{vmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}) & (a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}) \end{vmatrix}$$

Two matrices can be multiplied only if the number of columns of the first is equal to the number of rows of the second.

This value of l is sufficiently close to the trial value to make further iteration unprofitable unless the computation is carried to larger numbers of significant figures; it is, incidentally, in satisfactory agreement with the value obtained in the previous calculation (Art. 11.6).

The critical value of k may be determined by adding to l^2 the value of the axial buckling (1.580×10^{-4}), to obtain B_1^2 , and solving for k in Eq. (109).

12 LUMPED THERMAL ABSORBERS IN THE REACTOR

Many practical cases arise in which it is necessary to know the effect of an absorbing object of macroscopic size in the reactor. Typical cases are slugs for the production of radioactive isotopes, experimental apparatus for investigation of radiation effects, and control rods.

Usually the information desired is the rate of absorption of neutrons in the absorber for a given level of average neutron flux in the reactor. Often the effect of the absorber on reactivity is also desired. It is always true that if the absorber has macroscopic thermal absorption cross section Σ_a , and if the thermal neutron flux is ϕ , at some point in the absorber, then the rate of absorption per unit volume at that point is

$$\text{Rate of absorption} = \Sigma_a \phi, \quad \text{neutrons}/(\text{cm}^3)(\text{sec}) \quad (222)$$

Unless, however, the absorber is both small and only weakly absorbing, it cannot be assumed† that the flux at the absorber position is the same as that which would exist at that position if the absorber were absent. In general the absorber will depress the thermal flux in its vicinity, and the true rate of absorption can be determined only by a computation which evaluates the effect of this flux distribution.

A general treatment of the absorber problem is impractical here, for on the one hand, diffusion theory is inadequate for the treatment of very small (but highly absorbing) absorbers, while on the other hand, the treatment of large absorbers can become very complex if the geometry of the absorber-reactor combination lacks symmetry. Some relatively simple but useful specific cases are treated below.‡

12.1 Absorbers in Infinite Medium

The computation of absorbers of simple shapes in infinite diffusion media is relatively simple. For these cases the absorption rate is evaluated in terms of the thermal neutron flux in the medium infinitely far from the absorber. In many cases this computation can be used as an approximation for the more difficult case of the absorber in a finite reactor by methods discussed below.

To evaluate the absorption we consider the two-region problem, the absorber being region 1, the infinite diffusion medium being region 2, and the origin of coordinates being at the center of the absorber. There is a uniform source of thermal neutrons of strength S_1 neutrons/(cm³)(sec) in region 1 and a uniform source of strength S_2 neutrons/(cm³)(sec) in region 2. Practically, S_1 and S_2 are just the slowing-down densities into the thermal-neutron group in region 1 and region 2, respectively. For the infinite medium, since all neutrons slowed down are eventually absorbed, $S_2 = \phi_\infty \Sigma_{S2}$, where ϕ_∞ is the value of the thermal flux infinitely far from the absorber and Σ_{S2} is the macroscopic absorption cross section in region 2. It is assumed that the absorber is small enough that it does not produce a significant perturbation in the fast neutron density.§ If, then, for example, the slowing-down properties of the absorbing material are the same as those in region 2, $S_1 = S_2$. A

† The assumption is very poor if the absorber thickness is comparable to an absorption mean free path. If the thickness is equal to a small fraction ϵ of an absorption mean free path, then the assumption of no perturbation of flux by the absorber will lead to a fractional error of the same order of magnitude as ϵ .

‡ Many cases of thermally black cylindrical absorbers are treated in Ref. 12.

§ That is, its smallest dimension is a small fraction of a slowing-down length ($\sqrt{\tau}$) in the absorber material. If this condition is not met, the fast flux distribution must be taken into account by, for example, a two-group calculation (see Art. 11.22).

case of frequent occurrence is that in which the absorber is a material of much higher atomic weight than that of region 2. In such a case, S_1 may be taken to be zero.

Under the conditions outlined above, the thermal-neutron flux in both regions satisfies the equation (the group notation of Sec. 7 is used here)

$$D_s \nabla^2 \phi_s - \Sigma_s \phi_s + S = 0$$

$$\text{or} \quad \nabla^2 \phi_s - \kappa^2 \phi_s + \frac{S}{D_s} = 0 \quad (223)$$

where D_s and Σ_s are the diffusion constant and the absorption cross section, respectively, for thermal neutrons in the appropriate region and $\kappa^2 = \Sigma_s/D_s$.

For an absorber in the shape of an infinite slab of half thickness T or an infinitely long cylinder of radius R or of a sphere of radius R , the thermal fluxes ϕ_{s1} and ϕ_{s2} in regions 1 and 2, respectively, are given by equations of the form

$$\phi_{s1} = A_1 X + \frac{S_1}{\Sigma_{s1}} \quad (224)$$

$$\phi_{s2} = A_2 Y + \frac{S_2}{\Sigma_{s2}} \quad (225)$$

Table 17 gives the values of the constants A_1 and A_2 and the functions X and Y for the three geometrical cases.

Table 17. Flux Distributions for Absorbers in an Infinite Medium with Uniform Source of Thermal Neutrons

Type	Region 1	Region 2
	$\phi_{s1} = A_1 X + \frac{S_1}{\Sigma_{s1}}$	$\phi_{s2} = A_2 Y + \frac{S_2}{\Sigma_{s2}}$
Slab	$X = \frac{\cosh \kappa_{1z}}{\cosh \kappa_{1T}}$	$Y = \frac{\cosh \kappa_{2z} - \sinh \kappa_{2T}}{\cosh \kappa_{2T} - \sinh \kappa_{2T}}$
Cylinder	$I_0(\kappa_1 r)$	$K_0(\kappa_2 r)$
Sphere	$\frac{\sinh \kappa_1 r}{r}$	$\frac{\cosh \kappa_2 r - \sinh \kappa_2 r}{r}$
Slab	$A_1 = \frac{S_2/\Sigma_{s2} - S_1/\Sigma_{s1}}{\cosh \kappa_1 T + (D_{s1}\kappa_1/D_{s2}\kappa_2) \sinh \kappa_1 T}$	$A_2 = \frac{A_1 D_{s1} \kappa_1 \sinh \kappa_1 T}{D_{s2} \kappa_2 (\cosh \kappa_2 T - \sinh \kappa_2 T)}$
Cylinder	$\frac{S_2/\Sigma_{s2} - S_1/\Sigma_{s1}}{I_0(\kappa_1 R) + (D_{s1}\kappa_1/D_{s2}\kappa_2)[K_0(\kappa_2 R)/K_1(\kappa_2 R)]I_1(\kappa_1 R)}$	$\frac{A_1 D_{s1} \kappa_1 I_1(\kappa_1 R)}{D_{s2} \kappa_2 K_1(\kappa_2 R)}$
Sphere	$\frac{S_2/\Sigma_{s2} - S_1/\Sigma_{s1}}{(\sinh \kappa_1 R)/R + [D_{s1}(R\kappa_1 \cosh \kappa_1 R - \sinh \kappa_1 R)/D_{s2}R(1 + \kappa_2 R)]}$	$\frac{A_1 D_{s1}(R\kappa_1 \cosh \kappa_1 R - \sinh \kappa_1 R)}{D_{s2}(1 + \kappa_2 R)(\cosh \kappa_2 R - \sinh \kappa_2 R)}$

The relation for the flux in the absorber having been established from Table 18, the rate of absorption at any point in the absorber can be computed by Eq. (222). If, however, only the total rate of absorption by the absorber is desired, it may be evaluated by taking the derivative of the flux at the absorber surface and computing the density of neutron current into the absorber by Eq. (5):

$$J = -D \text{ grad } \phi \quad \text{neutrons}/(\text{cm}^2)(\text{sec})$$

For the one-dimensional cases considered in Table 17, grad ϕ is just $d\phi/dx$ for the slab and $d\phi/dr$ for the cylinder or sphere.

If the absorber is "black" to thermal neutrons, both Σ_{s1} and κ_1 in Table 17 become infinite. If the equations are used for such a case, the extrapolation distance ϵ should be applied (Art. 2.22). The effective boundary between regions 1 and 2 then lies at $T - \epsilon$ or $R - \epsilon$, as the case may be. For black boundaries of large curvature the

Table 18. Flux Distributions around Black Absorbers in an Infinite Medium with Uniform Source of Thermal Neutrons

Geometry of absorber	Thermal neutron flux, neutrons/(cm ²)(sec)	Current density into absorber surface, neutrons/(cm ²)(sec)
Infinite slab Half thickness = T Origin of coordinates at center	$\phi(x) = -\frac{S_2}{\Sigma_{a2}} \frac{(\cosh \kappa_2 x - \sinh \kappa_2 x)}{(\epsilon \kappa_2 + 1)(\cosh \kappa_2 T - \sinh \kappa_2 T)} + \frac{S_2}{\Sigma_{a2}}$ <p style="text-align: center;">or</p> $\phi(x) = -\frac{S_2}{\Sigma_{a2}} \frac{e^{-\kappa_2 x}}{(\epsilon \kappa_2 + 1)e^{-\kappa_2 T}} + \frac{S_2}{\Sigma_{a2}}$	$J(T) = -\frac{S_2}{\kappa_2(\epsilon \kappa_2 + 1)}$
Infinitely long cylinder Radius = R Origin of coordinates at center	$\phi(r) = -\frac{S_2}{\Sigma_{a2}} \frac{K_0(\kappa_2 r)}{K_0(\kappa_2 R) + \epsilon \kappa_2 K_1(\kappa_2 R)} + \frac{S_2}{\Sigma_{a2}}$	$J(R) = -\frac{S_2}{\kappa_2 \{ \epsilon \kappa_2 + [K_0(\kappa_2 R)/K_1(\kappa_2 R)] \}}$
Sphere Radius = R Origin of coordinates at center	$\phi(r) = -\frac{S_2 R}{\Sigma_{a2} r} \frac{\cosh \kappa_2 r - \sinh \kappa_2 r}{[\epsilon \kappa_2 + 1 + (\epsilon/R)](\cosh \kappa_2 R - \sinh \kappa_2 R)} + \frac{S_2}{\Sigma_{a2}}$ <p style="text-align: center;">or</p> $\bar{\phi}(r) = -\frac{S_2 R}{\Sigma_{a2} r} \frac{e^{-\kappa_2 r}}{[\epsilon \kappa_2 + 1 + (\epsilon/R)]e^{-\kappa_2 R}} + \frac{S_2}{\Sigma_{a2}}$	$J(R) = -\frac{S_2}{\kappa_2 [\epsilon \kappa_2 + (R\kappa_2/1 + R\kappa_2)]}$

more rigorous boundary condition is, however, ϕ_2'/ϕ_2 (boundary) = $1/\epsilon$. Table 18 gives the flux distribution in region 2 with this condition for the three absorber shapes considered. The table also gives the current density into the absorber surface.

It is evident from Table 17 that the effect of the absorber on the flux in medium 2 falls off rapidly with increasing x_2r (or r/L , where L is the diffusion length) and, even for black absorbers, becomes small at about three diffusion lengths from the absorber. The infinite-medium case may be used as a reasonable approximation to the case of an absorber in a finite reactor provided the undisturbed flux distribution in the reactor (i.e., the flux distribution before the absorber is installed) does not amount to a large flux variation over the region in which the absorber, when installed, will influence the flux distribution strongly. The infinite case is often used as a rough approximation even when this condition is not fulfilled. When the finite approximation is used, it is common practice to assume that the equivalent uniform source strength S_2 for the infinite case is just equal to the undisturbed slowing-down density at the point in the finite reactor where the absorber is to be installed. An alternate procedure, which makes some allowance for the leakage of thermal neutrons out of the reactor, is to evaluate an effective source strength by setting $S_2 = \phi \Sigma_{s2}$, where ϕ is the undisturbed thermal-neutron flux at the point where the absorber is to be installed. In either case a more accurate procedure (frequently not justified because of other inaccuracies) is to use, for the equivalent infinite value of S_2 , the average value obtained by averaging numerically the value of S_2 for the undisturbed finite reactor over a region extending two or three diffusion lengths around the point of installation of the absorber.

A rough estimate of the effect of the absorber on reactivity of the finite reactor can be derived from solution of the infinite case if the spatial perturbation relations are utilized (see Sec. 14). A possible procedure is as follows:

1. Evaluate the absorption rate a of the absorber. If the absorber is a sphere, a would be the total absorption rate; if the absorber is a cylinder, a would be the absorption rate per unit length.

2. Compute an equivalent thermal absorption cross section Σ_a , where

$$\Sigma_a = \frac{a}{\phi \times (\text{vol. of absorber over which } a \text{ is computed})} \quad (226)$$

and ϕ is the average thermal flux in the *undisturbed* reactor in the region where a is computed.

3. The absorber can then be treated, by perturbation theory, as an absorption cross section of magnitude Σ_a , distributed in the reactor over the volume occupied by the absorber.

12.2 Absorbers in Regular Array

Occasionally the case occurs in which identical absorbers are arranged in a regular or nearly regular array over the reactor or over some region of the reactor. For such a case the absorbers can be treated by the same methods as those used for calculation of the thermal utilization in a lumped reactor in Art. 9. First the region occupied by the absorbers is divided into cells symmetrical about the absorbers; the approximation is then made that the region occupied by the cells is infinite in extent. A "thermal utilization" f_{abs} is then computed for the absorbers in exactly the same way as the thermal utilization f was calculated for fuel lumps (Art. 9). Usually for these cases the total absorption of thermal neutrons by the absorbers is small compared with that of the other reactor materials. When this is the case, the relation can be written

$$\frac{\text{Absorption by all other materials (including fuel)}}{\text{Absorption by "lumped" absorbers}} = \frac{1}{f_{abs}} - 1$$

and an equivalent absorption cross section Σ_a can be assigned to the lumped absorber. The quantity Σ_a is that cross section which, if treated as uniformly distributed, will result in the same fractional neutron absorption as that of the lumped absorbers.

It is given by

$$\Sigma_e = \Sigma_{\text{reactor}} \left(\frac{1}{f_{\text{abs}}} - 1 \right) \quad (227)$$

where Σ_{reactor} is the total absorption cross section (or, in the case of "lumped" reactors, the effective absorption cross section) of all materials except the absorbers in question. After the effective cross section is determined, the absorbers may be treated as uniformly distributed, with absorption cross section Σ_e , for computation of both their total absorption and their effect on reactivity.

Frequently the region occupied by the array of absorbers does not extend over the entire reactor core but is symmetrically located in the core. In such a case, the properties of the region occupied by the array can be computed as indicated above, and the reactor may be computed as a symmetrical, multiregion problem, using the methods illustrated in Art. 7. Often the use of reflector savings will reduce such a problem to a two-region problem.

In the case of absorbers "black" to thermal neutrons the cell problem of most frequent occurrence is one which is symmetrical in cylindrical coordinates. In this case, the problem and its solution are stated as follows:

The absorber is a cylinder of radius R_1 ; the surrounding "cell" has radius R_2 . In the surrounding cell, there is a source density of thermal neutrons, assumed to be of uniform strength = q neutrons/(cm²)(sec). This source is just equal to the slowing-down density into the thermal energy group. The source strength is taken to be zero in the absorber. The boundary condition at the absorber surface is $(1/\phi)(d\phi/dr) = 1/\epsilon$, where ϕ is the thermal flux density and ϵ is the extrapolation distance (see Art. 2.22). The thermal flux ϕ in the cell outside the absorber is a function of the coordinate r , measured radially from the center line of the absorber, and follows the equation

$$\nabla^2 \phi - \kappa^2 \phi + \frac{q}{D} = 0$$

The solution is

$$\phi(r) = G[K_1(\kappa R_2)I_0(\kappa r) + I_1(\kappa R_2)K_0(\kappa r)] + \frac{q}{\Sigma_s}$$

where Σ_s is the macroscopic absorption cross section for thermal neutrons in the material surrounding the absorber and G is a constant whose value is given by

$$G = \frac{q/\Sigma_s}{\epsilon \kappa [K_1(\kappa R_2)I_1(\kappa R_1) - I_1(\kappa R_2)K_1(\kappa R_1)] - [K_1(\kappa R_2)I_0(\kappa R_1) - I_1(\kappa R_2)K_0(\kappa R_1)]}$$

The current density into the absorber is

$$J(R_1) = -\frac{D}{\epsilon} \frac{q}{\Sigma_s} \left[1 + \frac{1}{\frac{\epsilon \kappa [K_1(\kappa R_2)I_1(\kappa R_1) - I_1(\kappa R_2)K_1(\kappa R_1)]}{K_1(\kappa R_2)I_0(\kappa R_1) + I_1(\kappa R_2)K_0(\kappa R_1)} - 1} \right]$$

where D is the thermal-neutron diffusion constant in the material outside the absorber. The "thermal utilization" f_{abs} of the absorbers is given by

$$f_{\text{abs}} = \frac{2\pi R_1 J(R_1)}{q\pi(R_2^2 - R_1^2)}$$

12.3 Central Absorber in Bare Reactor

If the absorber is so shaped that the geometry of the absorber-reactor combination exhibits a high degree of symmetry, the problem can often be solved in as much detail as is desired by use of the group methods. The problem of this type which occurs most frequently is that of a central cylindrical absorber passing axially through a cylindrical reactor. The solution for such a system, treated by the two-group approximation, is given in Art. 7.5. The case of a bare reactor is considered. If a

reflected system is to be computed, the reflector saving can be computed first without the absorber and can be assumed to be unchanged by the presence of the absorber. This assumption is adequate unless the absorber diameter is an appreciable fraction of the reactor diameter.

The procedure is to solve the criticality problem for the reactor containing the absorber. The rate of absorption in the absorber can then be evaluated, if desired, from the thermal flux distribution which is obtained in arriving at the solution. The effect of the absorber on reactivity can be found by comparing the criticality condition (e.g., critical k_{∞}) of the reactor containing the absorber with that of the reactor without the absorber.

13 NONCRITICAL REACTORS

If the neutron balance in a reactor is such that exactly one of the neutrons emitted in a typical fission reacts with a fissionable atom to produce a subsequent fission, the reactor is said to be *critical*. If in a critical reactor there are no extraneous sources of neutrons, the neutron population will remain constant in time. This section deals with the behavior of reactors which are not critical. In general, the neutron population of such a reactor, if not zero, will vary with time (exception—subcritical reactor with source; see below).

13.1 General Concepts and Definitions

If the reactor, although not critical, is nearly so, the spatial distribution of neutrons in it will be very nearly the same as in the critical reactor. The general significance of quantities used to specify the characteristics of noncritical assemblies can best be appreciated in terms of such cases, which are also the ones occurring most frequently in practical problems. These quantities† k_{eff} , k_{ex} , and ρ are alternative descriptions of the relation between the number of neutrons in a given generation of the chain reaction and the number in the immediately succeeding generation. The effective multiplication factor k_{eff} is just the ratio of these two numbers:

$$k_{eff} = \frac{\text{No. neutrons born in } (n+1)\text{st generation}}{\text{No. neutrons born in } n\text{th generation}} \quad (228)$$

The excess multiplication factor k_{ex} is

$$\begin{aligned} k_{ex} = k_{eff} - 1 &= \frac{\left[\begin{array}{c} \text{No. neutrons born} \\ \text{in } (n+1)\text{st generation} \end{array} \right] - \left(\begin{array}{c} \text{No. born in} \\ n\text{th generation} \end{array} \right)}{\text{No. neutrons born in } n\text{th generation}} \\ &= \frac{\text{No. extra neutrons produced per generation}}{\text{No. neutrons starting the generation}} \end{aligned} \quad (229)$$

The reactivity ρ is

$$\rho = \frac{k_{ex}}{k_{eff}} \quad (230)$$

The value of k_{eff} in terms of the characteristic reactor quantities (Sec. 5) is

$$k_{eff} = \frac{k\bar{P}_{\infty}(E_s, B^2)}{\rho(1 + L^2B^2)} \quad (231)$$

where B^2 corresponds to the fundamental mode in the solution of the wave equation

$$\nabla^2\phi + B^2\phi = 0$$

for the reactor with proper boundary conditions. Equation (231) may be regarded as the definition of k_{eff} for reactors which can be described by an equation of that type, irrespective of whether or not the reactor is near criticality.

† The use of these symbols follows that of Weinberg and of Glasstone and Edlund. Other usages are extant.

Evidently, by Eq. (229), if an average time between generations of neutrons l_{avg} (the average generation time) could be specified (and if the spatial distribution of neutrons in the reactor were independent of time), the time dependence of neutron density n in the reactor would be given by

$$\begin{aligned} \frac{dn}{dt} &= n \frac{k_{ex}}{l_{avg}} \\ n &= n_0 \exp \frac{k_{ex}}{l_{avg}} t \end{aligned} \quad (232)$$

Because a fraction of the neutrons are *delayed* (i.e., emitted in the process of radioactive decay of fission products), it is possible to specify an average generation time only if k_{ex} is very small (about 0.0004 or less) and if k_{ex} has been constant for some time. In general, the time behavior of the neutron population is a complicated function of reactivity, the delayed-neutron characteristics, and the prompt neutron lifetime (see Art. 13.2 and Sec. 8-1).

If an extraneous source of neutrons, S neutrons/sec, is introduced into a subcritical reactor, the neutron flux in the reactor will, after a time, reach some steady value. In this condition the production of neutrons by the source plus fissions is just balanced by the loss of neutrons by leakage plus absorption. If the reactor is nearly critical, the number of neutrons produced per second is, by relation (228), S directly from the source plus $S k_{eff}$ from the immediately preceding generation plus $S(k_{eff})^2$ from the generation before that, etc.; i.e.,

$$\begin{aligned} \text{No. neutrons born per sec} &= S(1 + k_{eff} + k_{eff}^2 + k_{eff}^3 + \dots) \\ &= \frac{S}{1 - k_{eff}} \end{aligned} \quad (233)$$

In other words, the source production is increased by a factor $1/(1 - k_{eff})$ by the noncritical chain reaction. This factor $1/(1 - k_{eff})$ is sometimes called the *multiplication* M of the reactor. Strictly speaking, the source is multiplied by this exact factor only if the source is distributed throughout the reactor in the same way as the fissions are distributed in the critical reactor (i.e., according to the solution of the wave equation

$$\nabla^2 \phi + B^2 \phi = 0$$

However, if the reactor is not much subcritical, and if the source is located well inside the core, the above expression for the multiplication will hold to within a factor of 2 or 3.

13.2 Kinetic Behavior of a Bare, † Near-critical Reactor with Delayed Neutrons

The emission of delayed neutrons accompanies the β decay of a certain small group of fission products. The rate of emission by each member of the group of delayed emitters decreases exponentially with time after fission. There are six species of fission products which emit delayed neutrons in sufficient quantity to have an important effect on reactor kinetics. The fraction of *total fission neutrons* emitted by the i th member of the group of delayed emitters is designated by the symbol β_i , the decay constant of the emitter is designated by λ_i , and the concentration of the emitter per unit volume of reactor by C_i . Thus, if an instantaneous burst of N fission neutrons per unit volume of reactor were to occur at $t = 0$, the concentration C_i of the i th emitter species resulting from the burst would vary with time as

$$C_i = N \beta_i e^{-\lambda_i t} \quad (234)$$

and the rate of emission of delayed neutrons per unit volume by that emitter would

† Although the following discussion is based on bare reactor concepts, the results are applicable to simple reflected reactors provided an appropriate value is used for the effective neutron lifetime.

vary as

$$(\text{No. delayed neutrons of } i\text{th species emitted})/(\text{sec})(\text{unit volume}) = \lambda_i C_i = N \lambda_i \beta_i e^{-\lambda_i t} \quad (235)$$

Values of λ_i and β_i are tabulated for thermal-neutron fission of U^{235} in Tables 1 to 4 of Sec. 8-1 and Table 39 of Sec. 1-1. †

The number of *prompt* fission neutrons produced per unit volume per second in a reactor operating with a thermal-neutron flux density ϕ is $(k/p)\Sigma_a\phi(1-\beta)$, and if the concentrations C_i of all the delayed emitters are known, the rate of emission of delayed neutrons per unit volume can be specified as $\sum_i C_i \lambda_i$. If the reactor is

sufficiently near criticality that the spatial distribution of flux does not differ substantially from that which would exist in the *critical* reactor, the leakage of neutrons can be expressed in terms of the geometrical buckling B^2 of the critical reactor. ‡ If the assumption is made that the effective energy of the delayed neutrons is the same as that of the prompt ones, § the slowing-down density in the reactor will be

$$q = \left[\frac{k}{p} \Sigma_a \phi (1 - \beta) + \sum_i C_i \lambda_i \right] \bar{P}_\infty(E_s, B^2) \quad (236)$$

where $\bar{P}_\infty(E_s, B^2)$ is the Fourier transform of the infinite slowing-down kernel (see Art. 5.3). The balance equation which holds for thermal neutrons at every point in the reactor is

$$-DB^2\phi - \Sigma_a\phi + \left[\frac{k}{p} \Sigma_a \phi (1 - \beta) + \sum_i C_i \lambda_i \right] \bar{P}_\infty(E_s, B^2) = \frac{dn}{dt} = \frac{1}{v} \frac{d\phi}{dt} \quad (237)$$

where v is the average velocity of thermal neutrons. Applying the definition of k_{eff} [Eq. (231)] and the relation $L^2 = D/\Sigma_a$, Eq. (237) reduces to

$$[k_{eff}(1-\beta) - 1]\phi + p \frac{k_{eff}}{k\Sigma_a} \sum_i C_i \lambda_i = \frac{1}{(1 + L^2 B^2)v\Sigma_a} \frac{d\phi}{dt} \quad (238)$$

The quantity $1/[(1 + L^2 B^2)v\Sigma_a]$ is the effective lifetime of thermal neutrons in the finite reactor (see Art. 13.3) and is designated by the symbol l . To specify completely the variation of ϕ with time, Eq. (238) must be coupled with the six differential equations which specify the variation of the concentration C_i of the six delayed-neutron emitters. The complete set of equations becomes

$$\begin{aligned} [k_{eff}(1-\beta) - 1]\phi + \frac{pk_{eff}}{k\Sigma_a} \sum_i C_i \lambda_i &= l \frac{d\phi}{dt} \\ \frac{dC_i}{dt} &= -\lambda_i C_i + \frac{k}{p} \beta_i \Sigma_a \phi \end{aligned} \quad (239)$$

It is quite easy to obtain the solution of Eqs. (239) in the asymptotic case for which the reactor has been in a state of constant positive excess reactivity for an effectively infinite time. For this case ϕ and all the C_i are increasing exponentially with a constant period ¶ T , whose value is to be determined.

Assuming

$$\begin{aligned} \phi &= A e^{t/T} & C_i &= a_i e^{t/T} \\ \frac{d\phi}{dt} &= \frac{A}{T} e^{t/T} & \frac{dC_i}{dt} &= \frac{a_i}{T} e^{t/T} \end{aligned}$$

† See also Ref. 14. For delayed-neutron data on other isotopes, see Refs. 15 and 16. Table 5 of Sec. 1-1 lists other data contained in this handbook.

‡ Many practical reactor kinetics problems concern reactors which are not far from criticality, for which this approximation is valid. For the more general case, see Ref. 13.

§ This assumption should be examined critically for cases in which extreme accuracy is necessary, as, for example, when reactor periods are used for the measurement of reactivity changes.

¶ This result is to be expected on physical grounds; mathematically it results from the complete solution of Eqs. (239).

substitution in Eqs. (239) yields

$$a_i = \left[\frac{A}{(1/T) + \lambda_i} \right] \frac{k}{\rho} \beta \Sigma_a$$

and

$$(1 - \beta)k_{eff} - 1 + k_{eff} \sum_i \frac{\lambda_i \beta_i}{(1/T) + \lambda_i} = \frac{l}{T} \tag{240}$$

recognizing that $\beta = \sum_i \beta_i$, Eq. (240) can be reduced to

$$\frac{k_{eff} - 1}{k_{eff}} = \frac{l}{Tk_{eff}} + \sum_i \frac{\beta_i/T}{(1/T) + \lambda_i} \tag{241}$$

This equation, often called the *inhour equation*, relates the *asymptotic* or *stable* reactor period to the reactivity [note that $(k_{eff} - 1)/k_{eff} = \rho =$ reactivity]. It is a simple relation which is much used in experimental reactor physics for evaluating reactivity from the measured reactor period. Figure 20 gives T as a function of ρ for two values of the lifetime l . Also plotted is the curve for $l = 0$. Note that curves for other

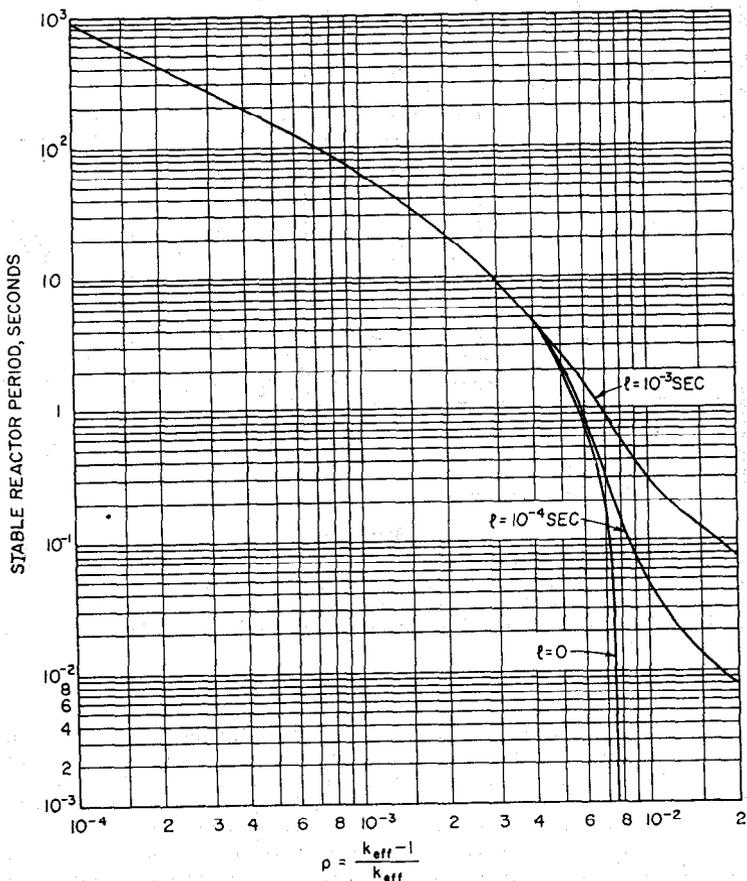


FIG. 20. The Inhour relation (delayed-neutron data).

values of l can be easily constructed from the latter curve simply by adding the term l/Tk_{eff} .

For values of ρ which are greater than β , the period becomes very short and the term $(1/T)$ becomes large compared with λ_i . When λ_i can be neglected, Eq. (241) reduces to

$$\rho = \frac{l}{Tk_{eff}} + \sum_i \beta_i = \frac{l}{Tk_{eff}} + \beta$$

or

$$T = \frac{l}{k_{eff}(\rho - \beta)} \quad (242)$$

This approximation is usually good for $\rho > (\beta + 0.0025)$.

In addition to the single positive value of T which will satisfy Eq. (241) (for a positive ρ), there are n negative values which will also satisfy the equation (n is the number of delayed-neutron groups). Thus the complete solution of Eqs. (239), for

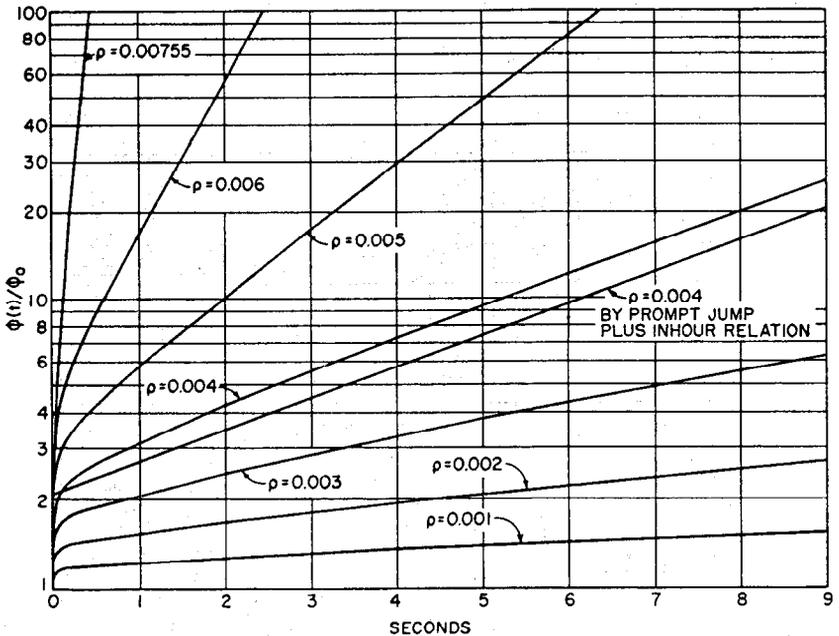


Fig. 21. Time variation of neutron flux $\phi(t)$ after instantaneous application of reactivity ρ to critical reactor operating at steady flux ϕ_0 . $l = 1 \times 10^{-4}$ sec.

the case in which ρ is constant after assuming some initial value at time $t = 0$, is a sum of exponentials:

$$\phi = A_0 e^{t/T_0} + A_1 e^{t/T_1} + \dots + A_n e^{t/T_n} \quad (243)$$

For ρ positive, T_0 and A_0 are positive and T_0 is equal to the asymptotic value discussed above; all the other A 's and T 's are negative. For ρ negative, T_0 and all the other T 's are negative and all the A 's are positive. This general solution is discussed at length in Sec. 8-1 and in texts on reactor theory (e.g., Glasstone and Edlund). The results for a particular value of l (10^{-4} sec) are plotted in Fig. 21.

Since the general solution of Eqs. (239) for any particular l involves considerable labor, it is useful to find approximations which can be used when only rough values are needed. A very crude approximation to the solution for positive instantaneous

changes in reactivity can be made in the following way: The problem is assumed to be that of determining the variation of flux in a reactor which has been operating for an effectively infinite time at constant flux ϕ_0 prior to the time t_0 , when the effective multiplication constant is increased instantaneously from the value 1 to the value $k_{eff} > 1$ and is maintained at that value for all time later than t_0 .

Just prior to the reactivity increase delayed neutrons were being formed at some steady rate such that their direct (unmultiplied) contribution to the flux ϕ_0 was $\phi_0\beta$. The value of the prompt effective multiplication constant was $(k_{eff})_{prompt} = 1 - \beta$. The prompt multiplication (M_{prompt}) was $1/[1 - (k_{eff})_{prompt}] = 1/\beta$. Thus the flux

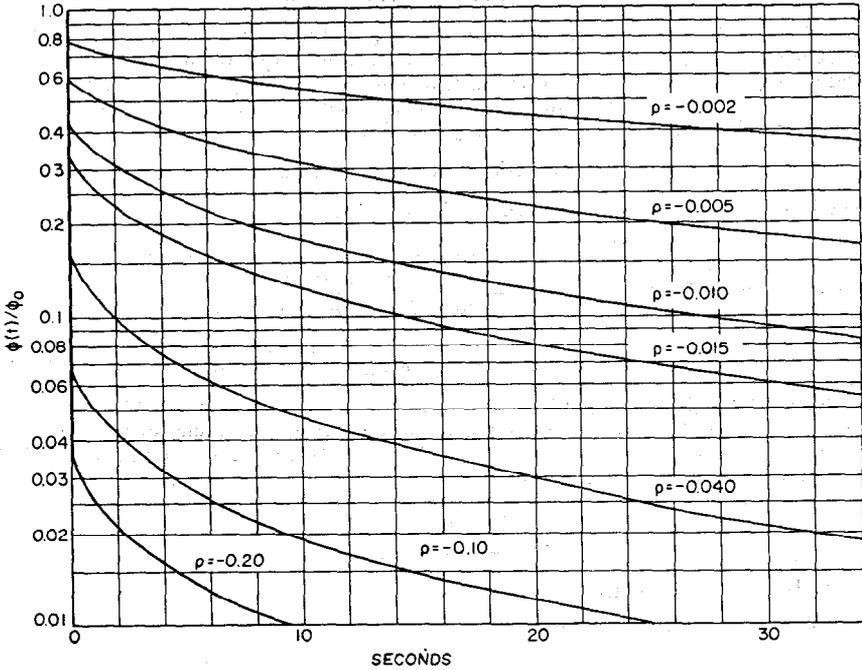


FIG. 22. Time variation of neutron flux $\phi(t)$ after instantaneous application of negative reactivity ρ to critical reactor operating at steady flux ϕ_0 . $l = 1 \times 10^{-4}$ sec.

ϕ_0 can be considered to arise from the prompt multiplication of the delayed-neutron source. This concept is illustrated by the identity

$$\phi_0 = \left(\begin{array}{l} \text{flux due directly} \\ \text{to delayed sources} \end{array} \right) \left(\begin{array}{l} \text{prompt} \\ \text{multiplication} \end{array} \right) = \phi_0\beta \frac{1}{\beta} = \phi_0$$

For a short time after the effective multiplication constant is increased to the new value k_{eff} , the flux due directly to delayed sources will remain nearly at the old value $\phi_0\beta$. The prompt multiplication will, however, have increased to the value

$$M_{prompt} = \frac{1}{1 - k_{eff}(1 - \beta)}$$

Consequently, the total flux will be expected to rise rapidly to a new value ϕ_1 , given by

$$\phi_1 = \frac{\phi_0\beta}{1 - k_{eff}(1 - \beta)} = \frac{\phi_0\beta}{\beta k_{eff} - k_{ex}} \tag{244}$$

Quite evidently the approximation does not apply if $k_{ex} > \beta$. For smaller values of k_{ex} a rough approximation to the kinetic behavior can be made by assuming that the flux rises instantaneously to the value given by Eq. (244) and thereafter follows the asymptotic variation, as given by Eq. (241). Such a curve is drawn in Fig. 21 for comparison with the correct curve.

The kinetic behavior of the neutron flux for negative values of reactivity is not strongly dependent on l after the first few seconds. Thus the curves for negative ρ in Fig. 22 can be used for other values of l except during the period of rapid flux decrease.

13.3 The Effective Neutron Lifetime

In the development of Eq. (238) the time required for neutrons to slow down was neglected; consequently the effective lifetime l which appears in the equation is really the effective lifetime of thermal neutrons l_{th} . The physical interpretation of the expression for l_{th} is as follows: The number of neutrons absorbed per unit volume per second, if the steady neutron density is n , is

$$\text{Absorption}/(\text{cm}^3)(\text{sec}) = n\nu\Sigma_a$$

The loss of thermal neutrons by leakage is [Eq. (102)]

$$\text{No. thermal neutrons leaking from } 1 \text{ cm}^3/\text{sec} = n\nu\Sigma_a(L^2B^2)$$

$$\begin{aligned} \text{Effective thermal-neutron lifetime} &= \frac{\text{neutron density}}{(\text{loss of neutrons})/(\text{cm}^3)(\text{sec})} \\ &= \frac{n}{n\nu\Sigma_a(1 + L^2B^2)} = \frac{1}{\nu\Sigma_a(1 + L^2B^2)} \quad (245) \end{aligned}$$

The true effective neutron lifetime is made up of the effective thermal lifetime plus the effective fast lifetime. Equation (254) is the correct expression for the effective lifetime in the two-group approximation. If the reactor is bare and if its properties are uniform, Eq. (254) reduces to

$$l = \frac{1}{\bar{v}_s k_p \Sigma_{as}} \frac{\phi_s^*}{\phi_f^*} + \frac{1}{\bar{v}_f k_p \Sigma_{af}} \frac{\phi_f}{\phi_s}$$

From Eqs. (92) and (252),

$$\frac{\phi_f}{\phi_s} = \frac{\Sigma_s + D_s B^2}{\Sigma_f} \quad \text{and} \quad \frac{\phi_s^*}{\phi_f^*} = \frac{\Sigma_f + D_f B^2}{\Sigma_f}$$

If the small difference between k and k_p is neglected and the reactor is near enough criticality that the characteristic equation (87) may be assumed to hold, the equation for l becomes

$$l = \frac{1}{\bar{v}_s \Sigma_{as}(1 + L_s^2 B^2)} + \frac{1}{\bar{v}_f \Sigma_{af}(1 + L_f^2 B^2)} \quad (246)$$

where \bar{v}_s is the average velocity of the thermal neutrons, \bar{v}_f is the average velocity of the fast neutrons, Σ_{as} is the macroscopic thermal absorption cross section, and Σ_{af} is the macroscopic "slowing-down cross section."

If the reactor is reflected by a material having a thermal-neutron lifetime much different from that of the core, the effective lifetime for the reactor may be significantly different from that for the equivalent bare reactor. Equation (254) may be used to determine the effective lifetime for this case. Since, however, the integration of the flux and adjoint functions over the reactor volume may be quite tedious, an alternate method is sometimes useful. The effective fast lifetime, which is usually a small fraction of the total lifetime, is taken to be that of the equivalent bare reactor, as given by the last term of Eq. (246), and the effective thermal lifetime is computed by a scheme devised by R. P. Feynman, which proceeds as follows: It is assumed that the criticality condition for the reactor has already been determined by the methods of

Art. 7 or 8 or by some other method. A small amount of $1/\nu$ absorber is then added uniformly throughout the reactor (core plus reflector). The average macroscopic absorption cross section due to this added poison is designated Σ_{ap} . The poisoned reactor is then solved by some convenient method (Art. 7 or 8), and its effective multiplication constant (k_{eff}) is determined. The quantity k_{eff} will, of course, be less than unity. Since the prompt neutron lifetime l is to be determined, it is assumed that all fission neutrons are prompt. Then, by Eq. (232), the rate of change of flux at any point in the poisoned reactor where the instantaneous value of flux is ϕ is

$$\frac{d\phi}{dt} = \frac{\phi(k_{eff} - 1)}{l}$$

Recalling, however, that the reactor would be critical (and flux constant) except for the presence of the added poison, one concludes that the rate of decrease of neutron density is just equal to the rate of absorption by the added poison and the rate of change of flux is

$$\frac{d\phi}{dt} = -\bar{\nu} \frac{dn}{dt} = -\bar{\nu} \phi \Sigma_{ap}$$

Equating the two rates of change of flux above yields

$$\frac{\phi(k_{eff} - 1)}{l} = -\bar{\nu} \phi \Sigma_{ap}$$

$$l = -\frac{k_{eff} - 1}{\bar{\nu} \Sigma_{ap}}$$

Or, more generally, if the uniform addition of thermal-neutron absorber of average macroscopic cross section Σ_{ap} results in a reactivity change Δk_{eff} , the prompt thermal-neutron lifetime l_{th} is

$$l_{th} = -\frac{\Delta k_{eff}}{\bar{\nu} \Sigma_{ap}} \quad (247)$$

Note that the above reasoning involves the tacit assumption that the addition of the thermal absorber does not change the flux distribution to an important extent. This can be true only if the amount of absorber added is quite small.

14 PERTURBATION RELATIONS

It is often useful to evaluate the effect on reactivity of small changes in the properties of an initially critical reactor. Perturbation theories have been developed which give approximate evaluations of such small changes, whether the changes be localized in space or distributed over the entire reactor. Perturbation treatments can be based on any of the group formulations of the reactor equation. Only specific formulas from the simpler treatments are given here. For general treatments and for the theory of the method other sources should be consulted (e.g., Refs. 7, 17, and 18).

14.1 First-order, Two-group Perturbation Formulas

The first-order, two-group perturbation theory was first published by L. W. Nordheim (CP-2824). This form of the theory is the most widely used in thermal reactor calculations. In the development of the formulas certain terms involving second orders of small quantities are neglected, but the major assumption is that the perturbation does not change significantly the flux distribution in the reactor. This condition can generally be used as a criterion of the applicability of the method.

In deriving the expression which evaluates the perturbation in terms of reactivity, the period of the perturbed reactor is first determined formally, and the period is then related to reactivity by means of the inhour equation [Eq. (241)]. Thus the starting point of the development is the time-dependent, two-group reactor equations

(the notation of Art. 7.1 is used here)

$$\begin{aligned} D_s \nabla^2 \phi_s - \Sigma_s \phi_s + \Sigma_f \phi_f &= \frac{1}{v_s} \frac{\partial \phi_s}{\partial t} \\ D_f \nabla^2 \phi_f - \Sigma_f \phi_f + k_p \Sigma_s \phi_s + \sum_i C_i \nu_i &= \frac{1}{v_f} \frac{\partial \phi_f}{\partial t} \\ \frac{\partial C_i}{\partial t} &= \beta_i k_p \sigma_s \phi_s - C_i \nu_i \end{aligned} \quad (248)$$

where k_p is the prompt infinite multiplication factor $\left[k = k_p \left(1 + \sum_i \beta_i \right) \right]$.

It is assumed that the solutions for the fluxes are separable into space-dependent and time-dependent terms, the latter being exponentials of reciprocal period ν . Equations (248) can then be written

$$\begin{aligned} D_s \nabla^2 \phi_s - \Sigma_s \phi_s + \Sigma_f \phi_f &= \frac{\nu}{v_s} \phi_s \\ D_f \nabla^2 \phi_f - \Sigma_f \phi_f + k_p \Sigma_s \phi_s &= \nu \left(\frac{\phi_f}{v_f} + k_p \Sigma_s \sum_i \phi_s \frac{\beta_i}{\nu + \nu_i} \right) \end{aligned} \quad (249)$$

The adjoint functions ϕ_f^* and ϕ_s^* are defined by the equations

$$\begin{aligned} D_s \nabla^2 \phi_s^* - \Sigma_s \phi_s^* + k_p \Sigma_s \phi_f^* &= \nu \left(\frac{\phi_s^*}{v_s} + k_p \Sigma_s \sum_i \phi_f^* \frac{\beta_i}{\nu + \nu_i} \right) \\ D_f \nabla^2 \phi_f^* - \Sigma_f \phi_f^* + \Sigma_f \phi_s^* &= \frac{\nu \phi_f^*}{v_f} \end{aligned} \quad (250)$$

No discussion of the general significance and properties of the adjoint functions will be given here; the approach will be taken that they are simply functions useful in the evaluation of perturbation effects. It can be shown that identical values of ν satisfy Eqs. (249) and (250) and that both equations define the same criticality condition ($\nu = 0$). The procedure for determining the effects of small perturbations on the critical reactor is as follows:

1. Write out the two-group equations for the fast and slow fluxes (ϕ_f and ϕ_s) in the critical reactor:

$$\begin{aligned} D_s \nabla^2 \phi_s - \Sigma_s \phi_s + \Sigma_f \phi_f &= 0 \\ D_f \nabla^2 \phi_f - \Sigma_f \phi_f + k \Sigma_s \phi_s &= 0 \end{aligned} \quad (251)$$

and the equations for the fast and slow adjoint functions (ϕ_f^* and ϕ_s^*) in the same critical reactor:

$$\begin{aligned} D_s \nabla^2 \phi_s^* - \Sigma_s \phi_s^* + k \Sigma_s \phi_f^* &= 0 \\ D_f \nabla^2 \phi_f^* - \Sigma_f \phi_f^* + \Sigma_f \phi_s^* &= 0 \end{aligned} \quad (252)$$

2. Solve the flux equations (251) for the critical values of the buckling B^2 by the methods of Arts. 7 or 8, and write out the solutions for the fast and slow fluxes as indicated in those articles.

3. Using the critical value of the buckling determined in 2 above, determine also the solutions for the fast and slow adjoint functions ϕ_f^* and ϕ_s^* . Note that Eqs. (252) are identical in form with Eqs. (251), the only difference being the changes in coefficients of the final terms. The methods of solution for ϕ_f^* and ϕ_s^* are therefore the same as those used in the solutions for ϕ_f and ϕ_s .

4. The values for the fluxes and adjoint functions having been obtained as functions of position over the entire volume of the reactor (core plus reflector), the effect on k_{eff} of local changes in any of the reactor properties can be evaluated in terms of these functions. The total effect on k_{eff} can be evaluated by integrating all the local perturbations over the entire volume of the reactor. The expression for the

total fractional change in k_{eff} , $\delta k_{eff}/k_{eff}$, is

$$\frac{\delta k_{eff}}{k_{eff}} = \frac{1}{\int_{\text{core volume}} k_p \Sigma_s \phi_s \phi_f^* dv} \left\{ \int_{\text{reactor volume}} [\phi_s \phi_f^* \delta(k\Sigma_s) - \phi_s \phi_s^* \delta\Sigma_s + (\phi_f \phi_s^* - \phi_f \phi_f^*) \delta\Sigma_f + \nabla \phi_s \nabla \phi_s^* \delta D_s + \nabla \phi_f \nabla \phi_f^* \delta D_f] dv \right\} \quad (253)$$

The quantity $\delta(k\Sigma_s)$ in the above equation may be written in either of two ways:

$$\delta(k\Sigma_s) = \delta(\eta\Sigma_{25}) = \Sigma_{25} \delta\eta + \eta \delta\Sigma_{25}$$

or

$$\delta(k\Sigma_s) = \Sigma_s \delta k + k \delta\Sigma_s$$

Note that since each term in both numerator and denominator of Eq. (253) contains a single product of a flux and an adjoint function, the absolute normalizations of both the flux and the adjoint are immaterial. The proper relationships must, of course, be maintained between ϕ_f and ϕ_s and between ϕ_s^* and ϕ_f^* .

14.2 The Neutron Lifetime

First-order, two-group perturbation theory gives the following expression for the neutron lifetime l^* in the finite reactor:

$$l^* = \frac{\int_{\text{reactor volume}} [(1/\bar{v}_s) \phi_s \phi_s^* + (1/\bar{v}_f) \phi_f \phi_f^*] dV}{\int_{\text{core volume}} k_p \Sigma_s \phi_s \phi_f^* dV} \quad (254)$$

where \bar{v}_s is the average velocity of a typical thermal neutron and \bar{v}_f is the average velocity of a typical neutron during its lifetime as a fast neutron. The quantity Σ_s is, of course, the average macroscopic absorption cross section for thermal neutrons.

14.3 Nonuniform Perturbations of a Homogeneous Bare Reactor

The spatial distributions of both the fluxes and the adjoint functions in a homogeneous bare reactor are given by the solutions of the wave equation

$$\nabla^2 \phi + B^2 \phi = 0$$

Solutions of the equation are given for the usual reactor shapes in Table 5. If the solution from column 4 of that table is taken as the solution for both $\phi_s(r)$ and $\phi_s^*(r)$, the solutions for $\phi_f(r)$ and $\phi_f^*(r)$ are simply

$$\phi_f(r) = \frac{B^2 D_s + \Sigma_s}{\Sigma_f} \phi_s(r) \quad (255)$$

and

$$\phi_f^*(r) = \frac{B^2 D_s + \Sigma_s}{k \Sigma_s} \phi_s^*(r) \quad (256)$$

With these expressions for the fluxes and adjoints the effects of local or distributed perturbations can be evaluated by Eq. (253).

14.4 Uniformly Distributed Perturbation of a Homogeneous Bare Reactor

For the case of perturbations distributed uniformly in a homogeneous bare reactor, Eq. (253) reduces to

$$\frac{\delta k_{eff}}{k_{eff}} = \frac{\delta(\eta\Sigma_{25})}{\eta\Sigma_{25}} - \frac{1}{1 + (D_s/\Sigma_s)B^2} \frac{\delta\Sigma_s}{\Sigma_s} + \frac{(D_f/\Sigma_f)B^2}{1 + (D_f/\Sigma_f)B^2} \frac{\delta\Sigma_f}{\Sigma_f} - \frac{(D_s/\Sigma_s)B^2}{1 + (D_s/\Sigma_s)B^2} \frac{\delta D_s}{D_s} - \frac{(D_f/\Sigma_f)B^2}{1 + (D_f/\Sigma_f)B^2} \frac{\delta D_f}{D_f} \quad (257)$$

The expression can, of course, be obtained also by differentiation of the characteristic equation

$$k_{eff} = \frac{\eta \Sigma_{25} / \Sigma_s}{[1 + (D_s / \Sigma_s) B^2][1 + (D_f / \Sigma_f) B^2]} \quad (258)$$

Frequently, for uniform perturbations, it is more convenient to use the expression derived from the more usual statement of the characteristic equation:

$$k_{eff} = \frac{k}{(1 + L^2 B^2)(1 + \tau B^2)} \quad (259)$$

The perturbation expression in terms of these quantities is

$$\frac{\delta k_{eff}}{k_{eff}} = \frac{\delta k}{k} - \frac{L^2 B^2}{1 + L^2 B^2} \frac{\delta L^2}{L^2} - \frac{\tau B^2}{1 + \tau B^2} \frac{\delta \tau}{\tau} - \left(\frac{L^2 B^2}{1 + L^2 B^2} + \frac{\tau B^2}{1 + \tau B^2} \right) \frac{\delta B^2}{B^2} \quad (260)$$

The above expressions can be used to evaluate the effects of uniform changes of core properties in reflected reactors provided these changes do not have a significant effect on reflector savings. It is usually safe to assume that changes in k alone will not produce important changes in reflector savings. Changes in B^2 will not cause large changes in reflector savings provided the reactor is already fairly large (say $k_\infty < 1.2$ for criticality). Changes in L^2 or τ will, in general, have important effects on reflector savings if they are both important in determining neutron leakage. If either L^2 or τ is small compared with $(L^2 + \tau)$, small changes in it will not have an important effect on reflector savings.

14.5 Perturbations in Lumped Reactors

The effects of perturbations in lumped reactors can be evaluated by the methods described in the foregoing parts of this section provided effective values for the perturbations over a typical cell of the reactor are used. These effective values may be determined by methods analogous to those described in Art. 9.

Often in lumped reactors the effects of perturbations on neutron leakage are not important, and the effect on k_{eff} can be evaluated in terms of changes in k alone, where k is expressed in terms of the four-factor formula

$$k = \eta \epsilon p f$$

When effects on leakage are unimportant,

$$\frac{\delta k_{eff}}{k_{eff}} = \frac{\delta k}{k} = \frac{\delta \eta}{\eta} + \frac{\delta \epsilon}{\epsilon} + \frac{\delta p}{p} + \frac{\delta f}{f} \quad (261)$$

The effects of the perturbation on η , ϵ , p , and f are determined by the methods applied in Sec. 9.

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